The Lascar Group and Reconstruction Problems in Model Theory

Mira Tartarotti

University of Oxford

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Abstract

Daniel Lascar proved in 1982 that an $\aleph_0$-categorical, G-finite theory T is determined up to bi-interpretation by its category of models $\mathbf{Mod}(T)$. The purpose of this dissertation is to identify the crucial underlying results of that theorem, so as to present the proof in a model-theoretic language. We isolate in particular one of the key concepts used in the proof of that theorem, which we will call Lascar-independence. We give examples and identify relevant properties of this independence notion.

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Chapter 1

Introduction

A common type of problem in mathematics is the following. Given a class of structures $\mathcal{S}$ (e.g. graphs, vector spaces etc.), an *invariant* is an assignment of objects $(I(S))_{S\in\mathcal{S}}$ such that whenever $S \simeq S'$, then $I(S) \simeq I(S')$. Here, $\simeq$ denotes some notion of equivalence that is suitable in the context. Examples of invariants are abundant - consider the the chromatic number of a graph, the dimension of a vector space, or the fundamental group of a topological space. Given such an invariant, how much does $I(S)$ *know* about the underlying structure S? In particular, which invariants $I(S)$ uniquely determine S up to equivalence? We say that $I(S)$ *reconstructs* a structure S, if whenever $I(S) \simeq I(S')$, then $S \simeq S'$. Out of the examples just mentioned, only the dimension of a vector space has this property.

Model Theory is also interested in such *reconstrction problems.* The suitable notion of equivalence in the context of first-order logic is that of a *bi-interpretation* between first-order theories/structures. (Henceforth we say *structure/theory* to refer to a structure/theory in the sense of first-order logic.) Loosely speaking, structures $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable if there are 'recipes' for translating expressions in $\mathcal{M}$ to expressions in $\mathcal{N}$ and vice versa, in a way that is in some sense inverse to one another. We say that a structure or theory is *reconstructed* by an invariant if it is determined by it up to bi-interpretation.
Two commonly considered invariants of a structure $\mathcal{M}$ are its endomorphism monoid $\mathrm{End}(\mathcal{M})$ and its automorphism group $\mathrm{Aut}(\mathcal{M})$. A priori, these are just algebraic structures, but one can turn them into a *topological monoid*, respectively a *topological group*, by equipping them with the *topology of pointwise convergence.* A result attributed by Ahlbrandt and Ziegler [2] to Coquand states that any $\aleph_0$-categorical structure is reconstructed by its topological automorphism group. When $\aleph_0$-categoricity is dropped, this does not hold anymore. Consider for example the structures $\langle \mathbb{Z}, +, 1\rangle$

and $\langle \mathbb{Z}, +, \cdot, 1\rangle$. Both have trivial automorphism group, but multiplication cannot be interpreted in the former.
A natural question to ask - again in the setting of $\aleph_0$-categoricity - is whether $\mathrm{Aut}(\mathcal{M})$ as a group, without any topology, still carries enough information to reconstruct $\mathcal{M}$. This was answered in the negative by Evans and Hewitt in 1990 [6]. Following this, it was an open question whether it would be enough to extend to $\mathrm{End}(\mathcal{M})$, which clearly knows at least as much about $\mathcal{M}$ as $\mathrm{Aut}(\mathcal{M})$. Bodirsky et al. [3] constructed in 2016 an example of two $\aleph_0$-categorical structures, whose endomorphism monoids are isomorphic as monoids, but not topological monoids. This proved that in general, an $\aleph_0$-categorical structure $\mathcal{M}$ cannot be reconstructed from $\mathrm{End}(\mathcal{M})$.

Makkai proved already in 1982 [18, Theorem 3.2] something that entails the Coquand-Ahlbrandt-Ziegler Theorem, but in a very different language - that of Grothendieck toposes. In the paragraphs following that result, Makkai states a problem, which entails the following. Given an $\aleph_0$-categorical theory T, denote by $\mathbf{Mod}(T)$ the category of its models, with elementary maps as morphisms. Can T be reconstructed by $\mathbf{Mod}(T)$? In other words, given a theory T' with $\mathbf{Mod}(T') \simeq \mathbf{Mod}(T)$, is T' necessarily bi-interpretable with T? In the $\aleph_0$-categorical setting, the category $\mathbf{Mod}(T)$ knows exactly as much about T as the endomorphism monoid $\mathrm{End}(\mathcal{M})$ of any countable model of T (see Theorem 4.11 and Proposition 4.5). Therefore, the counterexample of Bordirsky et al. answers Makkai's question in the negative. However, a paper by Lascar published in 1982 in response to Makkai shows that for a large class of theories, the answer is in fact 'yes' [16]. This is what we call the *Lascar Reconstruction Theorem*: An $\aleph_0$-categorical, *G-finite* theory T is determined up to bi-interpretation by the category $\mathbf{Mod}(T)$, or equivalently, by the endomorphism monoid $\mathbf{End}(\mathcal{M})$ of any of its countable models. Lascar defines in [16] an invariant of a theory T that can be viewed as a model-theoretic version of a Galois group, referred to today as the *Lascar group*, denoted $\mathrm{Gal}_{\mathrm{L}}(T)$. He shows that $\mathrm{Gal}_{\mathrm{L}}(T)$ can be turned into a topological group, and calls T *G-compact* if the relativized Lascar groups $\mathrm{Gal}_{\mathrm{L}}(T/X)$ are Hausdorff for all finite sets X. If additionally, the groups $\mathrm{Gal}_{\mathrm{L}}(T/X)$ are finite, then T is called *G-finite*.
In this dissertation, we isolate and examine the underlying ideas of the Lascar Reconstruction Theorem, and reformulate its proof in a way that is accessible to model theorists. As a part of this, we avoid the use of ultraproducts, using compactness arguments instead.

In Chapter 2 we introduce notation and recall the relevant model theoretic preliminaries. We briefly summarize the necessary definitions from category theory in Chapter 3.
Chapter 4 discusses the category of models $\mathbf{Mod}(T)$, as well as the endomorphism monoid and automorphism group of a structure. Proposition 4.8 shows that equivalences $\mathbf{Mod}(T) \to \mathbf{Mod}(T')$ between categories of models preserve directed colimits in a sense that will yield an important property in the proof of the Lascar Reconstruction Theorem - namely *model-continuity*.
We introduce in Section 5.1 the notions of an *interpretation* between structures, a *homotopy* between interpretations, and of a *bi-interpretations* of structures, following Ahlbrandt and Ziegler [2]. We avoid introducing interpretations between theories separately, by restricting to the $\aleph_0$-categorical case, where we can define them in terms of the unique countable model. In Section 5.2, we give a proof, following [2], of what we shall - given the historical background outlined above - refer to as the *Makkai-Coquand-Ahlbrandt-Ziegler Theorem*, or MCAZ.
Chapter 6 covers work done by Lascar [16] as well as Ziegler [27], defining the Lascar group of a theory in terms of a large (i.e. $> 2^{|T|}$) saturated model $\mathbb{U}$, and showing that this group is independent of the choice of such $\mathbb{U}$. Section 6.1 compares different approaches to defining the group of *Lascar-strong automorphisms* $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$. In Section 6.3 we give the proof, following Ziegler [27], that $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ is a topological group.
We discuss G-compactness, and its characterization in terms of the notions of *Shelah-*, *Kim-Pillay-* and *Lascar-strong types* in Section 7.1. Corollary 7.18 says approximately that in an $\aleph_0$-categorical, G-finite theory, the group $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U}/X)$ of Lascar-strong automorphisms over any finite set X contains a stabilizer subgroup $\mathrm{Aut}_Y(\mathbb{U})$ of another finite set Y. This is a crucial result for the proof of the Reconstruction Theorem.

One of the central theorems of Lascar [16] states that if two submodels $\mathcal{M}, \mathcal{N} \preccurlyeq \mathbb{U}$ are in a certain sense independent, then the union of their stabilizers $\mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ generates $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$ as a group. In Chapter 8, we formalize this independence relation, which we call *Lascar independence*. More specifically, we distinguish between *weak-* and *strong Lascar independence* (denoted $\mathop{\smile\!\!\!\!\!\mid}^{wL}$ and $\mathop{\smile\!\!\!\!\!\mid}^{sL}$), which corresponds to the distinction between *quasi-bounds* and *bounds*, as defined in [17, Chapter 6]. We restate the results of that chapter in terms of Lascar independence. In particular, we present proofs of the theorem mentioned above (Theorem 8.22), as well as Theorem 8.23, which serve as key tools in the proof of the Reconstruction Theorem. Moreover, we analyze $\mathop{\smile\hspace{-0.9em}|\hspace{0.2em}}^{wL}$ and $\mathop{\smile\hspace{-0.9em}|\hspace{0.2em}}^{sL}$ as independence relations, proving additional properties, and giving examples in some $\aleph_0$-categorical theories.

Finally in Chapter 9 we give the proof of the Lascar Reconstruction Theorem [16, Corollary 79], making use of the tools from the previous chapters. We isolate the key steps in Lascar's proof. Given an equivalence $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ we note first that by a result from Chapter 4, F is what we call *model-continuous*, i.e. preserves certain directed colimits. We then apply the main theorems of Chapter 8 on Lascar independence, to transition from model-continuity to another notion of continuity which we shall call *L-continuity*. Finally, we use the central result of Chapter 7, to show if T is G-finite, then L-continuity of F amounts to continuity w.r.t. pointwise convergence of the homomorphisms $F : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(F(\mathcal{M}))$. At this point, Lascar refers in his proof to the theorem by Makkai mentioned above, but we can in fact use the MCAZ Theorem as stated in Section 5.2, keeping this step purely model-theoretic in nature.

Chapter 2

Notation and Model Theoretic Preliminaries

Denote by $\mathcal{L}$ a first-order language, and by T an $\mathcal{L}$-theory. We will assume throughout the entire dissertation that theories are complete, and have infinite models. Moreover, we will assume that all structures are infinite.

We will denote $\mathcal{L}$-structures by caligraphic letters $\mathcal{M}, \mathcal{N}$ etc. We may occasionally distinguish between a structure $\mathcal{M}$ and its universe M, but we often ignore this distinction and simply write $\mathcal{M}$ for both the structure and the universe.
To denote that φ is a formula in the language $\mathcal{L}$, we sometimes simply write $\varphi \in \mathcal{L}$. We write $\varphi \in \mathcal{L}(n)$ when we want to specify that φ has n free variables, and $\varphi(v_1, \ldots, v_n)$ to highlight the free variables that appear in the formula. We say that φ is *closed* if it contains no free variables. If $\varphi(\overline{v})$ has free variables $v_{\lambda_1}, \ldots, v_{\lambda_n}$ for ordinals $\lambda_1, \ldots, \lambda_n$, and $\overline{x}$ is a tuple of length greater than $\max(\lambda_1, \ldots, \lambda_n)$, then we write $\varphi(\overline{x})$ for $\varphi(x_{i_1}, \ldots, x_{i_n})$.

Denote by $\mathrm{Th}(\mathcal{M})$ the theory of an $\mathcal{L}$-structure $\mathcal{M}$, that is, the set of all $\mathcal{L}$-sentences that hold in $\mathcal{M}$. For structures $\mathcal{M}, \mathcal{N}$ in the same language with $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$, we write $\mathcal{M} \equiv \mathcal{N}$ and say that $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*. When we want to highlight the underlying language $\mathcal{L}$, we may write $\mathrm{Th}^{\mathcal{L}}(\mathcal{M})$, and $\mathcal{M} \overset{\mathcal{L}}{\equiv} \mathcal{N}$.
If $\mathcal{M} \subseteq \mathcal{N}$ are structures in the same language, which agree on interpretations of closed terms, we say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$, and write $\mathcal{M} \leqslant \mathcal{N}$. If additionally, $\mathcal{M} \equiv \mathcal{N}$, then we write $\mathcal{M} \preccurlyeq \mathcal{N}$ and say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$. In this case, the inclusion map $\mathcal{M} \to \mathcal{N}$, which we denote by $\subseteq (\mathcal{M}, \mathcal{N})$, is an elementary map.

We recall here two fundamental results of Model Theory, namely the upward- and downward-version of the Löwenheim-Skoelem Theorem. Proofs of both these theorems can be found for example in Marker [20, Section 2.3].

Theorem 2.1 (Upward Löwenheim-Skolem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure, and $\kappa \geq \max(|M|, |\mathcal{L}|)$ a cardinal. Then there exists an elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$ with $|\mathcal{N}| = \kappa$.*

Theorem 2.2 (Downward Löwenheim-Skolem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure, and $X \subseteq \mathcal{M}$. Then there exists an elementary substructure $X \subseteq \mathcal{N} \preccurlyeq \mathcal{M}$ with $|\mathcal{N}| \leq \max(|X|, |\mathcal{L}|, \aleph_0)$.*

A major consequence of the Löwenheim-Skolem Theorems is that any theory which admits some infinite model, admits infinite models of all cardinalities $\kappa \geq |\mathcal{L}|$. Since this dissertation is concerned only with theories that have infinite models, we know arbitrarily large infinite models exist.

2.1 Elementary Maps and Automorphisms

We say that a map $f : \mathcal{M} \to \mathcal{N}$ between two $\mathcal{L}$-structures is an *embedding* if it is injective and preserves the interpretation of relation-, function-, and constant-symbols in $\mathcal{L}$. We say that f is an *elementary embedding* (or just *elementary*) if it preserves all $\mathcal{L}$-formulas, i.e. for any $\varphi \in \mathcal{L}(n)$ and $\overline{x} \in \mathcal{M}^n$,

$$\mathcal{M} \models \varphi(\overline{x}) \iff \mathcal{N} \models \varphi(f(\overline{x})).$$

Note that any elementary map must be injective, since we fix equality as a symbol in every language. If additionally f is surjective, then we say that f is an *isomorphism*, and write $\mathcal{M} \simeq \mathcal{N}$.
We denote the set of elementary maps from a model $\mathcal{M}$ into itself by $\mathrm{End}(\mathcal{M})$. Note that $\mathrm{End}(\mathcal{M})$ forms a monoid under composition, called the *endomorphism monoid* of $\mathcal{M}$. We denote the set of automorphisms (i.e. bijective elementary maps) of $\mathcal{M}$ into itself by $\mathrm{Aut}(\mathcal{M})$. This forms a group, called the *automorphism group* of $\mathcal{M}$. For any $X \subseteq \mathcal{M}$, write $\mathrm{Aut}_X(\mathcal{M})$ for the subgroup of automorphisms which fix X pointwise.

For any set A, the symmetric group $\mathrm{Sym}(A)$ carries the *topology of pointwise convergence*, i.e. the topology generated by the sets

$$\{f \in \mathrm{Sym}(A) \mid f(a) = b\} \text{ where } a, b \in A.$$

The name is justified by the fact that $(f_n)_{n\in\omega}$ converges to f in $\mathrm{Sym}(A)$ w.r.t. this topology if and only if it converges pointwise w.r.t. the discrete topology on A, that is, if for any $a \in A$, $f_n(a) = f(a)$ for large enough $n \in \omega$.
Note that $\mathrm{Aut}(\mathcal{M})$ is closed in $\mathrm{Sym}(M)$ under the topology of pointwise convergence: If $(f_n)_{n\in\omega}$ is a sequence in $\mathrm{Aut}(\mathcal{M})$ which converges to some $f \in \mathrm{Sym}(M)$, then for any $\varphi \in \mathcal{L}(k)$ and $\overline{x} \in \mathcal{M}^k$, find $n \in \omega$ with $f(\overline{x}) = f_n(\overline{x})$, and so

$$\mathcal{M} \models \varphi(\overline{x}) \iff \mathcal{M} \models \varphi(f_n(\overline{x})) \iff \mathcal{M} \models \varphi(f(\overline{x})).$$

Moreover, it is not difficult to see that $\mathrm{Sym}(M)$ forms a topological group under the topology of pointwise convergence. That is, composition and inversion are continuous. $\mathrm{Aut}(\mathcal{M})$ is clearly a subgroup, and thus a topological group under the topology of pointwise convergence, itself.

2.2 Types, Saturation, Homogeneity and Universality

We recall in this section the notion of a *type* in a theory or structure. Moreover, we give definitions of *saturation*, *homogeneity* and *universality* of a structure, and state some basic results about how these properties relate to one-another. The section is based on Marker [20, Chapter 4], which we refer to for the proofs of several of the results mentioned, as well as additional material.

Fix a language $\mathcal{L}$, a complete $\mathcal{L}$-theory T, and an $\mathcal{L}$-structure $\mathcal{M}$.

Definition 2.3. For a cardinal κ, a *partial κ-type* of T is a collection p of formulas in free variables $(v_\lambda)_{\lambda\in\kappa}$ such that $p \cup T$ is consistent.
We call a partial type p *complete* (or simply *type*) if for any formula φ in free variables $(v_\lambda)_{\lambda\in\kappa}$, we have $\varphi \in p$ or $\neg\varphi \in p$.

Note that if p is complete, then it is necessarily closed under conjunction, so by Compactness, $p \cup T$ is consistent if every $\varphi \in p$ is consistent with T.
We denote the set of κ-types of T by $S_\kappa(T)$. $S_\kappa(T)$ carries a topology generated by the basic open sets

$$\{p \in S_\kappa(T) \mid \varphi \in p\} \text{ where } \varphi \in \mathcal{L}(\kappa).$$

We call this the κ-th *Stone Space*. We denote set of all types on T by $S(T)$. The Compactness Theorem tells us that $S_\kappa(T)$ is compact.

Definition 2.4. We say that a type $p \in S(T)$ is *isolated* if there exists a formula $\varphi \in p$ such that for all other $\psi \in p$,

$$T \vdash \varphi \to \psi.$$

In other words, a type p is isolated if it contains a formula that no other type contains, or equivalently, if the singleton $\{p\}$ is open in $S_\kappa(T)$.
We can similarly define types on structures. For any set X, write $\mathcal{L}_X$ for the expansion of the language $\mathcal{L}$ by new constant symbols c_x for each $x \in X$.

Definition 2.5. For $X \subseteq \mathcal{M}$, a κ-type of $\mathcal{M}$ over X is a κ-type in the $\mathcal{L}_X$-theory $\mathrm{Th}^{\mathcal{L}_X}(\mathcal{M})$ of $\mathcal{M}$. We denote the set of κ-types over X in $\mathcal{M}$ by $S_\kappa(\mathcal{M}/X)$.

Definition 2.6. For $X \subseteq M$ and any κ-tuple $\overline{a} \in \mathcal{M}^\kappa$, the *type of $\overline{a}$ over X* in $\mathcal{M}$ is defined as

$$\mathrm{tp}^{\mathcal{M}}(\overline{a}/X) := \big\{\varphi(\overline{v}) \mid \varphi \in \mathcal{L}_X(\kappa) \text{ s.t. } \mathcal{M} \models \varphi(\overline{a})\big\}.$$

We call the type of $\overline{a}$ over the empty set, the type of $\overline{x}$in $\mathcal{M}$, and denote it by $\mathrm{tp}^{\mathcal{M}}(\overline{a})$. When the underlying structure $\mathcal{M}$ is clear from the context, we simply write $\mathrm{tp}(\overline{a}/X)$.

Note that $\mathrm{tp}^{\mathcal{M}}(\overline{a}/X)$ is a complete type in the $\mathcal{L}_X$-theory of $\mathcal{M}$.

When we consider types of sets $A \subseteq \mathcal{M}$, we need to specify an enumeration $\overline{A} = (a_\lambda)_{\lambda<\kappa}$ in order to make it clear which variable should be interpreted by which element of the set. In practice, we will simply assume that some enumeration is present and fixed, and write $\mathrm{tp}(\overline{A})$, even when A is a priori just a set.

Definition 2.7. We say that $\mathcal{M}$ *realizes* a type $p(\overline{v})$ if there exists a tuple $\overline{a} \subseteq \mathcal{M}$ with $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = p$. For such a tuple $\overline{a}$, we write $\mathcal{M} \models p(\overline{a})$.

The types in $S_\kappa(\mathcal{M}/X)$ need not necessarily be realized in $\mathcal{M}$. As an example, the set of formulas

$$\big\{v \geq n \mid n \in \mathbb{N}\big\}$$

is not realized in $\mathbb{R}$, and can be completed to a type $p \in S_1(\langle \mathbb{R}, < \rangle/\mathbb{N})$. Structures which do realize all types over all sets are called *saturated.*

Definition 2.8. For any cardinal κ, we say that a structure $\mathcal{M}$ is *κ-saturated* if for any set $X \subseteq \mathcal{M}$ with $|X| < \kappa$, all 1-types $p \in S_1(\mathcal{M}/X)$ over X are realized in $\mathcal{M}$. We say $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

Lemma 2.9. *Let $\mathcal{M}$ be κ-saturated. Then for any $X \subseteq M$ with $|X| < \kappa$ and $\lambda < \kappa$, $\mathcal{M}$ realizes all λ-types of T over X.*

Proof. We prove this by induction on λ. The base case ($\lambda = 1$) follows immediately by assumption.
Assume now that $\lambda < \kappa$ and that $\mathcal{M}$ satisfies all λ-types over sets $|X| < \kappa$. Let now $p \in S_{\lambda+1}(\mathcal{M}/X)$ for some $|X| < \kappa$. Define p' as the subset of p containing all formulas $\varphi \in p$ in which the variable $v_{\lambda+1}$ does not appear freely. By the inductive assumption, there exists $\overline{x} \in M^\lambda$ that satisfies p'. Now, let $X' := X \cup \overline{x}$. Then $|X'| \leq \max(|X|, \lambda) < \kappa$. For each $\varphi \in p \setminus p'$, we may replace any occurrence of free variables in $(v_\mu)_{\mu\in\lambda}$ by the respective element of the tuple $(x_\mu)_{\mu\in\lambda}$. We obtain a complete 1-type over X', which must be realized by some $x_{\lambda+1} \in M$ by assumption. Thus, we obtain a tuple $(x_\mu)_{\mu<\lambda+1}$ which satisfies p.
Suppose now that $\gamma < \kappa$ is a limit ordinal, and for any $\lambda < \gamma$ and $|X| < \kappa$, all λ-types over X are realized in $\mathcal{M}$. Then, let $p \in S_\gamma(\mathcal{M}/X)$ for some set $X \subseteq M$ with $|X| < \kappa$. We can find by iterative application of the argument used for the successor step for each $\lambda < \gamma$ a tuple $\overline{x}^\lambda$ that satisfies the formulas in p which only contain free variables $(v_\mu)_{\mu<\lambda}$, in such a way that $\overline{x}^\mu \subseteq \overline{x}^\lambda$ for each $\mu < \lambda < \gamma$. Then, $\overline{x} := \bigcup_{\lambda<\gamma} \overline{x}^\lambda$ realizes p. □

We now come to the notion of *homogeneity* and *universality* of a structure. What we define below as a *homogeneous* structure is often called *strongly homogeneous* in the literature (see for example [10]).

Definition 2.10. For an $\mathcal{L}$-structure $\mathcal{M}$ and infinite cardinal κ, we say that $\mathcal{M}$ is

- *κ-homogeneous* if for any $\overline{a}, \overline{b} \in \mathcal{M}^{<\kappa}$ with $\mathrm{tp}^{\mathcal{M}}(\overline{a}) = \mathrm{tp}^{\mathcal{M}}(\overline{b})$, there exists $f \in \mathrm{Aut}(\mathcal{M})$ with $f(\overline{a}) = \overline{b}$;
- *κ-universal* if for any model $\mathcal{N} \models T$ with $|\mathcal{N}| < \kappa$, there is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$.

We say $\mathcal{M}$ is *homogeneous* if it is $|M|$-homogeneous. We say that $\mathcal{M}$ is *universal* if it is $|M|^+$-universal.

The following is Corollary 4.3.19 in Marker [20]. (Note our definition of κ-homogeneity is stronger than the one in Marker, but by Proposition 4.2.13 of the same reference, these are equivalent for $\kappa = |M|$.)

Theorem 2.11. *For an infinite structure $\mathcal{M}$, the following are equivalent.*

(i) $\mathcal{M}$ is saturated.

(ii) $\mathcal{M}$ is homogeneous and universal.

Any theory has at most one saturated structure of any given cardinality, up to isomorphism. This is Theorem 4.3.20 in Marker [20].

Theorem 2.12. *If $\mathcal{M}$ and $\mathcal{N}$ are saturated structures in a language $\mathcal{L}$, with $\mathcal{M} \equiv \mathcal{N}$ and $|M| = |N|$, then $\mathcal{M} \simeq \mathcal{N}$.*

Remark 2.13. *Clearly, since isomorphisms preserve types, they also preserve saturation, so if $\mathcal{M}$ is saturated and $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{N}$ is saturated*

To conclude this section, we prove a Lemma that we will need in certain technical parts of the proof of Lascar's Reconstruction Theorem, specifically in Section 7.2 and Chapter 8.

Lemma 2.14. *Let $\mathcal{M} \preccurlyeq \mathcal{N}$ be saturated models of an $\mathcal{L}$-theory T, such that $\kappa := |\mathcal{M}|$ is a regular cardinal and $|\mathcal{L}| \leq \kappa < |N|$. Moreover, let $A \subseteq M$ with $|A| < \kappa$, and*

$$f_1, \ldots, f_n \in \mathrm{Aut}(\mathcal{N}).$$

Then there exists a saturated structure $A \subseteq \mathcal{M}' \preccurlyeq \mathcal{N}$ with $|M'| = \kappa$ such that $f_i|_{\mathcal{M}'} \in \mathrm{Aut}(\mathcal{M}')$ for all $i = 1, \ldots, n$.

Proof. Assume w.l.o.g. that $n = 1$, and let $f := f_1$. The proof is analogous for $n > 1$. We will construct a chain of length κ of saturated models

$$A \subseteq \mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{N},$$

all of which are saturated, and of cardinality κ, such that for all $\mu < \lambda < \kappa$,

$$\mathcal{M}_\lambda \supseteq \bigcup_{k \in \mathbb{Z}} f^k(\mathcal{M}_\mu).$$

<u>Claim:</u> Such a chain $(\mathcal{M}_\lambda)_{\lambda<\kappa}$ exists.

Recall that since $\mathcal{M}$ is saturated, it is κ^+-universal by Theorem 2.11. That is, each model of size $\leq \kappa$ embeds elementarily into $\mathcal{M}$. We shall define $(\mathcal{M}_\lambda)_{\lambda<\kappa}$ inductively. To construct $\mathcal{M}_0$, notice first that

$$\left| \bigcup_{k \in \mathbb{Z}} f^k(A) \right| = \max(\aleph_0, |A|) \leq \kappa$$

so by downward Löwenheim-Skolem, we can find a model $\mathcal{M}' \preccurlyeq \mathcal{N}$ containing $\bigcup_{k \in \mathbb{Z}} f^k(A)$, of size κ. By universality, $\mathcal{M}'$ embeds elementarily into $\mathcal{M}$, so there exists a model $\mathcal{M}_0 \simeq \mathcal{M}$ such that $\mathcal{M}' \preccurlyeq \mathcal{M}_0$. Since $\mathcal{M}$ is saturated, $\mathcal{M}_0$ is saturated by Remark 2.13.

Assume now that for fixed $\lambda < \kappa$, models $\mathcal{M}_0, \ldots, \mathcal{M}_\lambda$ have already been constructed with the desired properties. We construct $\mathcal{M}_{\lambda+1}$ as follows: Again by downward Löwenheim-Skolem, and the fact that

$$\left| \bigcup_{k \in \mathbb{Z}} f^k(\mathcal{M}_\lambda) \right| = \max(\aleph_0, |M_\lambda|) = \kappa.$$

we can find a saturated model $\mathcal{M}_{\lambda+1} \preccurlyeq \mathcal{N}$ containing $\bigcup_{k \in \mathbb{Z}} f^k(\mathcal{M}_\lambda)$, with $|\mathcal{M}_{\lambda+1}| = \kappa$. At limit steps $\delta \in \mathit{Lim} \cap \kappa$, we just define

$$\mathcal{M}_\delta := \bigcup_{\lambda < \delta} \mathcal{M}_\lambda.$$

It is easy to see that $\mathcal{M}_\delta$ is again saturated, of size $|M|$, and contains $\bigcup_{k \in \mathbb{Z}} f^k(\mathcal{M}_\lambda)$ for all $\lambda < \delta$. This finishes the proof of the claim.

Having constructed $(\mathcal{M}_\lambda)_{\lambda < \kappa}$ as in the claim, we set $\mathcal{M}' := \bigcup_{\lambda < \kappa} \mathcal{M}_\lambda \subseteq \mathcal{N}$, which forms a model of T in a canonical way (see Chapter 4). Some basic cardinal arithmetic yields $|\mathcal{M}'| = \kappa \cdot \kappa = \kappa$. Moreover, $\mathcal{M}'$ is saturated: For any $X \subseteq \mathcal{M}'$ with $|X| < \kappa$, we can find for each $x \in X$ some $\lambda(x) < \kappa$ with $x \in \mathcal{M}_{\lambda(x)}$, and since κ is regular, we get

$$\lambda^* := \max(\lambda(x) \mid x \in X) < \kappa.$$

Then by saturation of $\mathcal{M}_{\lambda^*}$, any type over X is realized in $\mathcal{M}_{\lambda^*}$, and therefore in $\mathcal{M}'$. Finally, note that $\mathcal{M}'$ is closed under f and f^{-1} by construction: If $x \in \mathcal{M}_\lambda$ for some $\lambda < \kappa$, then $f(x), f^{-1}(x) \in \mathcal{M}_{\lambda+1} \subseteq \mathcal{M}'$. $\square$

2.3 $\aleph_0$-Categoricity

Large parts of this dissertation focus on $\aleph_0$-*categorical* theories and structures. The two central reconstruction theorems discussed - namely the MCAZ Theorem (5.11) and the Lascar Reconstruction Theorem (9.16) - are results about $\aleph_0$-categorical theories, and fail when this assumption is dropped.

Definition 2.15. An $\mathcal{L}$-theory T is called $\aleph_0$-*categorical* if all its countable models are isomorphic. An $\mathcal{L}$-structure $\mathcal{M}$ is called $\aleph_0$-*categorical* if $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical.

The famous Ryll-Nardzewski Theorem gives several equivalent characterizations of $\aleph_0$-categoricity, both in terms of the automorphism group of a countable model, and the Stone Spaces $S_n(T)$ of the theory.
Before we state this theorem, recall the definition of an *oligomorphic* group action.

Definition 2.16. An action of a group G on a set A is called *oligomorphic* if for each $n \in \omega$ the set of orbits on A^n

$$\{G \cdot (a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A\}$$

is finite.

See for example [10, Theorem 7.3.1] for a reference of the following.

Theorem 2.17 (Ryll-Nardzewski). *Let T be a theory in a countable language $\mathcal{L}$. Then the following are equivalent.*

(i) T is $\aleph_0$-categorical.

(ii) The Stone Space $S_n(T)$ is finite, for each $n \in \omega$.

(iii) Each type in $S_n(T)$ is isolated, for each $n \in \omega$.

(iv) The automorphism group of some countable model of T is oligomorphic.

2.4 Model Completeness and Quantifier Elimination

We will assume throughout most of this dissertation that the theories we are working with are *model-complete*, or more specifically, admit *quantifier elimination*. We briefly introduce these concepts and why they are useful, as well as justify assuming them in the context of reconstruction problems. This section is based on [10, Section 2.6 and 2.7]

Definition 2.18. We say that a theory T is *model-complete* if for any two models $\mathcal{M}, \mathcal{N} \models T$, every embedding $f : \mathcal{M} \to \mathcal{N}$ is elementary.

Note that in particular, if T is model-complete, then for any models $\mathcal{M} \leqslant \mathcal{N}$ we get $\mathcal{M} \preccurlyeq \mathcal{N}$, since the inclusion map $\subseteq : \mathcal{M} \to \mathcal{N}$ is an embedding.

Definition 2.19. We say that an $\mathcal{L}$-theory T admits *quantifier elimination* (or QE) if for any $\mathcal{L}$-formula $\varphi(\overline{v})$, there exists a quantifier-free $\mathcal{L}$-formula $\psi(\overline{v})$ such that

$$T \vdash \forall \overline{v} : \ \varphi(\overline{v}) \leftrightarrow \psi(\overline{v}).$$

We say that an $\mathcal{L}$-structure $\mathcal{M}$ admits QE if its theory does.
Since embeddings preserve quantifier-free formulas, any theory that admits QE is model-complete.

Given a language $\mathcal{L}$, there is a canonical way of extending it to a language $\mathcal{L}^+ \supseteq \mathcal{L}$ such that any $\mathcal{L}$-theory T can be expanded uniquely to an $\mathcal{L}^+$-theory T^+ which admits QE and is *bi-interpretable* with T (see Definition 5.8). Namely, for each formula $\varphi \in \mathcal{L}(n)$, we add a new n-ary relation symbol R_φ to $\mathcal{L}$, and let $\mathcal{L}^+$ be the resulting language. Then, given an $\mathcal{L}$-theory T, we add for each $\varphi \in \mathcal{L}(n)$ the $\mathcal{L}^+$-formula

$$\forall \overline{v} : \ \varphi(\overline{v}) \leftrightarrow R_\varphi(\overline{v})$$

to T, and obtain an $\mathcal{L}^+$-theory T^+.
This construction is often referred to in the literature as *Morleyisation*. Hodges [10] calls it *atomisation*.
We shall assume throughout Chapter 4 and Chapters 6-9, that any theory we work with has QE, and thus, is model-complete. Therefore when we say that a model is a *substructure* of another model, we will in fact mean that it is an *elementary substructure*. To justify this, note that in an $\aleph_0$-categorical theory T, the corresponding atomisation T^+ is clearly still $\aleph_0$-categorical, and T and T^+ are bi-interpretable, in the sense of Definition 5.8. Therefore, assuming QE does not reduce generality of the statement of the Lascar Reconstruction Theorem.

2.5 Elimination of Imaginaries

This section gives a short introduction to *imaginaries*, as introduced by Bruno Poizat. The contents are based on [10, Section 4.3]. They will not be used in later chapters, and are intended only as an aside.

Let $\mathcal{M}$ be a structure in some language $\mathcal{L}$, and T its $\mathcal{L}$-theory. For a formula $\varphi(\overline{v}, \overline{w}) \in \mathcal{L}(2n)$, we say that φ is an *equivalence formula* on T if T proves that φ defines an equivalence relation, i.e. $T \vdash \forall \overline{v}\ \varphi(\overline{v}, \overline{v})$ and so on. In other words, φ is an equivalence formula if $E := \varphi(\mathcal{M}) \subseteq \mathcal{M}^n \times \mathcal{M}^n$ is an equivalence relation on $\mathcal{M}^n$. In this case, we call E a *definable equivalence relation* on $\mathcal{M}$.

Definition 2.20. An *imaginary element* of $\mathcal{M}$ is an equivalence class $[\overline{x}]_E$ of a definable equivalence relation E.

Note that if E is a definable relation which is an equivalence on a definable set $X \subseteq \mathcal{M}^n$, then we can simply extend E to $E' := (E \cap X^2) \cup (\mathcal{M}^n \setminus X)^2$, which is a definable global equivalence.

Example 2.21. In any structure $\mathcal{M}$, all elements $x \in \mathcal{M}$ can be identified with the imaginary element $[x]_=$ of the quotient of the definable equivalence relation $\mathcal{M}^2 = \{(x,y) \in \mathcal{M}^2 | x = y\}$. Moreover, for any $\mathcal{L}$-formula $\varphi(\overline{v})$, the set $\varphi(\mathcal{M}^n)$ of tuples satisfying φ, as well as its complement, are imaginary elements, under the definable equivalence relation $E(\overline{v}, \overline{w}) :\iff \varphi(\overline{v}) \leftrightarrow \varphi(\overline{w})$. That is to say, all definable sets are imaginary elements.

It can be useful to be able to 'access' imaginary elements from the structure $\mathcal{M}$, via quotient maps. When quotient maps of definable equivalence relations are definable in the structure $\mathcal{M}$, we say that $\mathcal{M}$ has *elimination of imaginaries* (or simply, $\mathcal{M}$ *eliminates imaginaries*). Note that the definition we give here corresponds to what Hodges calls *uniform elimination of imaginaries* in [10].

Definition 2.22. We say that the structure $\mathcal{M}$ *eliminates imaginaries* if for any definable equivalence relation E on $\mathcal{M}^n$ there exists $k \in \omega$, and a definable function $f_E : \mathcal{M}^n \to \mathcal{M}^k$ such that for any $\overline{x}, \overline{y} \in \mathcal{M}^n$,

$$E(\overline{x}, \overline{y}) \iff f_E(\overline{x}) = f_E(\overline{y}).$$

The notion of elimination of imaginaries was introduced by Poizat [23] in 1983. It was originally defined as a property of a theory, rather than a model. We will say that a theory T eliminates imaginaries if all its models (or equivalently one of its models) eliminate imaginaries.
We immediately see that T eliminates imaginaries if and only if for every formula $\varphi(\overline{u}, \overline{v})$ such that T proves that φ is an equivalence relation, there exists another formula $\psi(\overline{v}, \overline{w})$ such that

$$T \vdash \forall \overline{u} \exists! \overline{w} \forall \overline{v} : \ (\varphi(\overline{u}, \overline{v}) \leftrightarrow \psi(\overline{v}, \overline{w}))$$

which was the original definition by Poizat.
One example of a theory which eliminates imaginaries is ACF_p - the theory of algebraically closed fields of characteristic p, see Poizat [23, Section 3]). This is not a trivial fact. We will give a much simpler example here.

Example 2.23. The structure $\langle \mathbb{Z}, \mathbb{N} \rangle$ consisting of the integers, with a unary relation that is interpreted as $\mathbb{N}$, does not eliminate imaginaries. Consider the definable equivalence relation $E(x, y) :\equiv x \in \mathbb{N} \leftrightarrow y \in \mathbb{N}$. The only definable, non-empty sets in this structure are products of the sets $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z} \setminus \mathbb{N}$, so there clearly does not exist a definable function $f : \mathbb{Z} \to \mathbb{Z}^k$ such that

$$(x \in \mathbb{N} \leftrightarrow y \in \mathbb{N}) \iff (f(x) = f(y)).$$

If we add the usual order $\leq$ to our structure, then such a quotient map can be defined, namely

$$f := \big\{(x, y) \in \mathbb{Z}^2 \;\big|\; (x, y \in \mathbb{N} \wedge \forall z \in \mathbb{N}\ y \leq z) \ \vee \ (x, y \notin \mathbb{N} \wedge \forall z \notin \mathbb{N}\ z \leq y)\big\}.$$

This is simply the function that maps all elements of $\mathbb{N}$ to 0 and all elements outside of $\mathbb{N}$ to -1. Adding the relation $\leq$ made it possible to define these two elements.
In a more general sense, an infinite set M with only a unary relation $R \subseteq M$ (where $R \neq \emptyset$ and $R \neq \mathcal{M}$) does not eliminate imaginaries, but by adding a constant in R and one outside of R, we obtain a structure that does eliminate imaginaries.

2.6 Many-Sorted Structures and the Eq-Construction

Even before the first literal mention of elimination of imaginaries, Saharon Shelah presented in [26] a canonical way of assigning to each structure $\mathcal{M}$ a structure $\mathcal{M}^{eq}$ which eliminates imaginaries and is bi-interpretable with $\mathcal{M}$. Poizat [23] calls the so-called *eq-construction* an "invention of ingenious simplicity" (translated by the author). This section may be skipped, as it is only relevant to the concluding remarks in Chapter 9.

Although $\mathcal{M}^{eq}$ was originally defined as a one-sorted structure, it is syntactically more comfortable to move to a many-sorted language. This also has the advantage that $\mathcal{M}^{eq}$ will be interpretable in $\mathcal{M}$ (see Proposition 5.9). We will begin this section by giving a brief introduction to many-sorted first-order logic.
Recall that in one-sorted first-order logic, a language consists of predicate symbols, function symbols, and constant symbols. An interpretation of such a language consists of one universe M, and interpretations of relations/functions/constants on M
The use of a one-sorted language becomes impractical when we want to look at structures which contain different 'sorts' of elements with entirely different roles. One example of this is the class of vector spaces. The elements of vector spaces are vectors, but there is another set of objects which makes up the data of a vector

space, namely the scalars. In the one-sorted language of vector spaces, we are unable to express statements about scalars. The problem is that vectors and scalars are entirely different objects, which ought to be treated differently inside the vector space structure. A solution to this is using a two-sorted language.

A κ-sorted language consists of a set S of κ-many *sorts*, as well as function symbols, relation symbols, and constant symbols. An arity now not just specifies the number of inputs a function/relation, but also the sorts which the different inputs ought to come from. Similarly, a constant must be of one fixed sort. An interpretation of such a language consists of universes $(M_s)_{s\in S}$ (one for each sort), as well as interpretations of the symbols, with suitable arities. Moreover, there are separate variables for each sort $s \in S$.

In the example of vector spaces, the suitable signature consists of two sorts $S = \{Vectors, Scalars\}$. This allows us to not only express statements about addition on $Vectors$, and scalar multiplication on $Scalars \times Vectors$, but also about addition and multiplication on $Scalars$. We are therefore able to express first-order statements about the field of scalars, which would not have been possible in the first-order solution, where scalars only appeared in the form of indices of unary function symbols.

Let us now fix a (one-sorted) structure $\mathcal{M}$ with $T := \mathrm{Th}(\mathcal{M})$, and construct the many-sorted language $\mathcal{L}^{eq}$, as well as the $\mathcal{L}^{eq}_T$-structure $\mathcal{M}^{eq}$.

Definition 2.24. For each definable equivalence relation E on $\mathcal{M}^n$, let S_E be a sort, and let f_E be a function symbol with arity $(S_=)^n \to S_E$. We add all symbols of the language $\mathcal{L}$ to the language $\mathcal{L}^{eq}_T$, on the sort $S_=$.

We interpret $S_=$ in $\mathcal{M}^{eq}$ as $\mathcal{M}$, with the $\mathcal{L}$-structure of $\mathcal{M}$, and interpret S_E as $\mathcal{M}/E$. The function symbol f_E is interpreted in $\mathcal{M}^{eq}$ as

$$f_E : \begin{cases} \mathcal{M}^n \to \mathcal{M}/E \\ \overline{x} \mapsto [\overline{x}]_E. \end{cases}$$

Finally, we define T^{eq} to be the $\mathcal{L}^{eq}_T$-theory of $\mathcal{M}^{eq}$.

Note that if $\mathcal{M}, \mathcal{M}'$ are $\mathcal{L}$-structures with $\mathcal{M} \equiv \mathcal{M}'$, then any $\mathcal{L}$-formula $\varphi(\overline{v}, \overline{w})$ defines an equivalence on $\mathcal{M}$ if an only if it defines an equivalence on $\mathcal{M}'$. Therefore, the notation $\mathcal{L}^{eq}_T$ is justified. Moreover, we get the following.

Lemma 2.25. *If $\mathcal{M} \overset{\mathcal{L}}{\equiv} \mathcal{N}$, then $\mathcal{M}^{eq} \equiv \mathcal{N}^{eq}$, i.e. $\mathcal{M}^{eq}$ and $\mathcal{N}^{eq}$ are elementarily equivalent as $\mathcal{L}^{eq}_T$-structures, where $T := \mathrm{Th}(\mathcal{M})$.*

Proof Sketch. Formally, one would prove this by symbolic induction on sentences. We give here a brief argument that can easily be formalized.
We notice that the only symbols in $\mathcal{L}_T^{eq}$ which involve sorts other than $S_=$ are the quotient maps f_E. Moreover, we can replace any quantifier over a sort S_E with a quantifier over the sort $S_=$, to obtain

$$\mathcal{M}^{eq} \models \forall v_1^E \exists v_2^E : \varphi(v_1^E, v_2^E) \leftrightarrow \forall v_1 \exists v_2 : \varphi(f_E(v_1), f_E(v_2)),$$

where v_i^E are variables of sort S_E, and v_i are variables of sort $S_=$. Now, recall that $f_E(\overline{v}) = f_E(\overline{w})$ is a definable equivalence relation on $\mathcal{M}$, and $\mathcal{N}$. Therefore, we can reduce any sentence that holds in $\mathcal{M}^{eq}$ to a sentence just on $\mathcal{M}_=$, and analogously for $\mathcal{N}^{eq}$. $\square$

By a similar argument, we see that $\mathcal{M}^{eq}$ indeed eliminates imaginaries. Again, we will only give a brief sketch of how to prove this.

Proposition 2.26. *For any structure $\mathcal{M}$, $\mathcal{M}^{eq}$ eliminates imaginaries.*

Proof Sketch. This holds by a similar agument as the one above. By construction, all imaginary elements of $\mathcal{L}$-definable equivalence relations on $S_=$ are eliminated in $\mathcal{M}^{eq}$. But in fact, these are all the definable equivalence relations on $\mathcal{M}^{eq}$. $\square$

Chapter 3

Category Theoretic Preliminaries

The purpose of this chapter is to provide the necessary category-theoretic background for the remainder of this dissertation. We recall the notions of functors, natural transformations, and equivalence of categories. The definitions in the following are adopted from [24], to which we refer the reader for a more detailed introduction to category theory.

Recall that a category $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$ of *objects*, as well as the set of *morphisms* $\mathrm{Hom}(x, y)$ between any two objects x and y. Additionally, one can *compose* morphisms of appropriate (co)domains, and for each object x there is a morphism $1_x : x \to x$ which acts as an identity w.r.t. composition.

We say that objects x and y are *isomorphic* in $\mathcal{C}$ if there exist morphisms $f : x \to y$ and $g : y \to x$ such that $g \circ f = \mathrm{id}_x$ and $f \circ g = \mathrm{id}_y$.
The most basic example of a category is **Set**, which consists of all sets as objects, and functions as morphisms. A slightly more interesting example is the category of groups **Grp**, whose objects are groups, and morphisms between groups are homomorphisms.

Definition 3.1. For categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ consists of

- A map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,
- For every pair $(x, y) \in \mathrm{Ob}(\mathcal{C})$, a map $F : \mathrm{Hom}^{\mathcal{C}}(x, y) \to \mathrm{Hom}^{\mathcal{D}}(Fx, Fy)$,

such that

(i) $F(\mathrm{id}_x) = \mathrm{id}_{Fx}$ for each $x \in \mathrm{Ob}(\mathcal{C})$,

(ii) $F(g \circ f) = F(g) \circ F(f)$ for any $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}^{\mathcal{C}}(x, y)$, $g \in \mathrm{Hom}^{\mathcal{C}}(y, z)$.

Going back to the examples above, there exists a canonical functor from **Grp** to **Set**, which maps each group to its underlying set, and each homomorphism $G \to G$ to itself. In other words, this functor simply *forgets* the algebraic structure of groups, so that only a set remains. Such functors are called *forgetful*.

Definition 3.2. For functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* from F to G, denoted by $\eta : F \Rightarrow G$, consists morphisms $\eta_x : F(x) \to G(x)$ in $\mathcal{D}$ for each $x \in \mathrm{Ob}(\mathcal{C})$, such that for any $x, y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}^{\mathcal{C}}(x, y)$, the following diagram commutes.

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd}$$

We say that η is a *natural isomorphism* if additionally, the morphism $\eta_x : F(x) \to G(x)$ is an isomorphism in $\mathcal{D}$.

It is easy to see that two functors (with suitable co-/domains) can be composed to a new functor. Moreover, the identity $\mathrm{id}_{\mathcal{C}}$ on any category $\mathcal{C}$ is a functor. With this in mind, we define what it means for categories to be equivalent.

Definition 3.3. An *equivalence* of categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $\eta : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\epsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$.
We sometimes call the functor F itself an equivalence of categories.

We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, and write $\mathcal{C} \simeq \mathcal{D}$, if there exists an equivalence between them. It is not difficult to see that $\simeq$ is in fact an equivalence relation. One can sensibly say that equivalences preserve essentially all properties of categories that are of interest in category theory. This justifies using equivalence of categories as the notion of 'sameness' which we shall use for categories. If, for example, $\mathcal{C}(T)$ and $\mathcal{D}(T)$ are categories associated with a first-order theory, then we say that we can *reconstruct* or *know* $\mathcal{C}(T)$ from $\mathcal{D}(T)$, if $\mathcal{D}(T)$ already determines $\mathcal{C}(T)$ up to equivalence.

The following properties of a functor will give a useful alternative characterization of equivalence of categories.

Definition 3.4. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is

- *full* if the map $F : \mathrm{Hom}^{\mathcal{C}}(x, y) \to \mathrm{Hom}^{\mathcal{D}}(Fx, Fy)$ is surjective for all $x, y \in \mathrm{Ob}(\mathcal{C})$;

- *faithful* if the map $F : \mathrm{Hom}^{\mathcal{C}}(x, y) \to \mathrm{Hom}^{\mathcal{D}}(Fx, Fy)$ is injective for all $x, y \in \mathrm{Ob}(\mathcal{C})$;
- *essentially surjective* if for each object $y \in \mathrm{Ob}(\mathcal{D})$, there exists an object $x \in \mathrm{Ob}(\mathcal{C})$ such that $F(x)$ is isomorphic to y in $\mathcal{D}$.

If a functor is full and faithful, we simply call it *fully faithful* (f.f.).

We will refer to [24, Theorem 1.5.9] for a proof of the following basic result in category theory.

Proposition 3.5. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Example 3.6. Recall the forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$. With Proposition 3.5 in mind, it is clear that F is not an equivalence, since F is not full.

Example 3.7. Consider the category $\mathbf{Grp}^*$, obtained by factoring out $\mathbf{Grp}$ by isomorphism. Clearly, the two categories are equivalent via

$$F : \begin{cases} \mathbf{Grp}^* \to \mathbf{Grp} \\ G \mapsto G \\ f \mapsto f. \end{cases}$$

Chapter 4

The Category of Models

Lascar's Reconstruction Theorem states that under certain conditions, the category of models of T determines the theory T up to bi-interpretability. What we mean by that is that if T' is another theory with an equivalent category of models, then it is bi-interpretable with T'. In this chapter, we define the category of models of T, denoted by $\mathbf{Mod}(T)$, and state some of its properties. In particular, we are interested in the properties of functors between categories of models. We prove for example that equivalences $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ in some sense preserve unions of elementary chains (see Proposition 4.8), and map countable models to countable models, as well as preserve $\aleph_0$-categoricity (Corollary 4.9). These results will be useful in Section 9.1.

For an $\aleph_0$-categorical theory T, we will write $\mathcal{M}_T$ to denote 'the unique countable model' of T. This is practical, but a slight abuse of notation, since $\mathcal{M}_T$ is only unique up to non-unique isomorphism. Then we associate with an $\aleph_0$-categorical theory T the endomorphism monoid $\mathbf{End}(\mathcal{M}_T)$ of $\mathcal{M}_T$, which is a category, with only one object. We will see that in the $\aleph_0$-categorical case, $\mathbf{End}(\mathcal{M}_T)$ carries all the information that $\mathbf{Mod}(T)$ does - it *reconstructs* $\mathbf{Mod}(T)$.
Similarly, we can consider the automorphism group $\mathrm{Aut}(\mathcal{M}_T)$ as a category, that consists of a single object, with morphisms $\mathrm{Aut}(\mathcal{M}_T)$. We denote this category by $\mathbf{Aut}(\mathcal{M})$.

Definition 4.1. For a theory T, we denote by $\mathbf{Mod}(T)$ the category where

- objects are models of T, i.e. $\mathrm{Ob}(\mathbf{Mod}(T)) = \mathrm{Mod}(T)$;
- morphisms between models are elementary maps, i.e. $\mathrm{Hom}(\mathcal{M}, \mathcal{N}) = \{f : \mathcal{M} \to \mathcal{N} \mid f \text{ is elementary}\}$;
- composition is regular function composition, and identities are identity maps.

We will call an object of a category which admits at least one morphism into all other objects, a *source* of the category.

Lemma 4.2. *Essentially surjective functors map sources to sources.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be essentially surjective, and $x \in \mathrm{Ob}(\mathcal{C})$ a source. Now, let $y \in \mathrm{Ob}(\mathcal{D})$, and find some $y' \in \mathcal{C}$ such that $F(y')$ is isomorphic to y via some isomorphism $f : F(y') \to y$ in $\mathcal{D}$. Since x is a source, we can find a morphism $g : x \to y'$ in $\mathcal{C}$. Then $f \circ F(g) : F(x) \to y$ is a morphism in $\mathcal{D}$. $\square$

Remark 4.3. *Recall that we say a model $\mathcal{M}$ of T is prime if it embeds elementarily into all other models. This is equivalent to saying that $\mathcal{M}$ is a source in the category of $\boldsymbol{Mod}(T)$.*

Corollary 4.4. *Equivalences $\boldsymbol{Mod}(T) \to \boldsymbol{Mod}(T')$ map prime models to prime models.*

Proof. This follows immediately from Lemma 4.2 and Remark 4.3. $\square$

If T and T' are $\aleph_0$-categorical, Corollary 4.4 tells us that an equivalence maps countable models to countable models. From this, it follows that $\mathbf{Mod}(T)$ reconstructs $\mathbf{End}(\mathcal{M}_T)$, and $\mathbf{Aut}(\mathcal{M}_T)$.

Proposition 4.5. *If T, T' are $\aleph_0$-categorical theories with $\boldsymbol{Mod}(T) \simeq \boldsymbol{Mod}(T')$, then $\boldsymbol{End}(\mathcal{M}_T) \simeq \boldsymbol{End}(\mathcal{M}_{T'})$ and $\boldsymbol{Aut}(\mathcal{M}_T) \simeq \boldsymbol{Aut}(\mathcal{M}_{T'})$ as groups.*

Proof. Let $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ be an equivalence of categories. By Corollary 4.4 and $\aleph_0$-categoricity, $F(\mathcal{M}_T)$ is a countable model of T', so there exists an isomorphism $\iota : F(\mathcal{M}_T) \to \mathcal{M}_{T'}$. Now, it is easy to see that the map $G : \mathbf{End}(\mathcal{M}_T) \to \mathbf{End}(\mathcal{M}_{T'})$ which maps $f \in \mathrm{End}(\mathcal{M}_T)$ to

$$G(f) := \iota \circ f \circ \iota^{-1},$$

is a monoid-isomorphism. Equivalently, G is a fully faithful and essentially surjective functor from $\mathbf{End}(\mathcal{M}_T)$ to $\mathbf{End}(\mathcal{M}_{T'})$.
The same functor G can be restricted to $\mathbf{Aut}(\mathcal{M}_T)$, and gives an equivalence $\mathbf{Aut}(\mathcal{M}_T) \simeq \mathbf{Aut}(\mathcal{M}_{T'})$. $\square$

Remark 4.6. *In general, it is clear that from $\boldsymbol{End}(\mathcal{M})$, we can reconstruct $\boldsymbol{Aut}(\mathcal{M})$, by taking those morphisms in $\boldsymbol{End}(\mathcal{M})$ which are invertible.*

We have seen that if T and T' are $\aleph_0$-categorical, then equivalences $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ map countable models to countable models. This is true in general - as long as T and T' are theories in a countable language. This follows essentially from the downward Löwenheim-Skolem Theorem, and the fact that essentially surjective f.f. functors preserve (co)limits.

Definition 4.7. We say that a sequence $(\mathcal{M}_\lambda)_{\lambda<\kappa}$ of models of some theory T is an *elementary chain* if for all $\mu < \lambda < \kappa$ we have

$$\mathcal{M}_\mu \preccurlyeq \mathcal{M}_\lambda.$$

We say that the chain is *strict*, if $\mathcal{M}_\mu \neq \mathcal{M}_\lambda$ for all $\mu < \lambda < \kappa$.

If $(\mathcal{M}_\lambda)_{\lambda<\kappa}$ is an elementary chain of length κ, of models of T, then we can canonically construct a limit structure $\bigcup_{\lambda<\kappa} \mathcal{M}_\lambda$ on the union of the universes M_λ. Interpret relation- and function-symbols on $\bigcup_\lambda \mathcal{M}_\lambda$ as the union of the interpretations on $\mathcal{M}_\lambda$. Because the chain is elementary, functionality is preserved. Finally, for a constant symbol c, we see by elementarity of the chain (and transfinite induction) that c must be interpreted as the same element in each of the $\mathcal{M}_\lambda$, so we can interpret c in $\bigcup_\lambda \mathcal{M}_\lambda$ as that element. Write $\mathcal{M}^* := \bigcup_\lambda \mathcal{M}_\lambda$.
It is easy to check that $\mathcal{M}_\lambda \preccurlyeq \mathcal{M}^*$ for all $\lambda < \kappa$. Denote by $\subseteq_\lambda$ the inclusion map $\mathcal{M}_\lambda \to \mathcal{M}^*$. Then $\mathcal{M}^*$ satisfies the following *universal property*: Firstly, if we denote by $\subseteq_{\alpha,\alpha+1}$ the inclusion map $\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$, then the following diagram obviously commutes.

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "\subseteq_{0,1}"] \arrow[dr, "\subseteq_0"'] & \mathcal{M}_1 \arrow[d, "\subseteq_1"'] & \cdots & \mathcal{M}_\alpha \arrow[dll, "\subseteq_\alpha"] & \cdots \\
 & \mathcal{M}^* & \cdots & & \cdots
\end{tikzcd}$$

Secondly, if $\mathcal{N} \models T$ is another model, with elementary maps $\iota_\lambda : \mathcal{M}_\lambda \to \mathcal{N}$ such that a diagram like the one above commutes - i.e. $(\iota_{\lambda+1})|_{\mathcal{M}_\lambda} = \iota_\lambda$ for all $\lambda < \kappa$ - then $\iota^* := \bigcup_\lambda \iota_\lambda$ is the unique elementary map from $\mathcal{M}^*$ to $\mathcal{N}$ such that $\iota^*|_{\mathcal{M}_\lambda} = \iota_\lambda$ for all $\lambda < \kappa$, or equivalently, such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{M}_\lambda \arrow[rr, "\subseteq"] \arrow[dr, "\subseteq"'] \arrow[ddr, bend right, "\iota_\lambda"'] & & \mathcal{M}_{\lambda+1} \arrow[dl, "\subseteq"] \arrow[ddl, bend left, "\iota_{\lambda+1}"] \\
 & \mathcal{M}^* \arrow[d, dashed, "\iota^*"] & \\
 & \mathcal{N} &
\end{tikzcd}$$

This construction is a special case of what category theorists call a *directed colimit*: If $(I, \leq)$ is a *directed set* (i.e. a partial order where for each $i, j \in I$ there exists $k \geq i, j$) and $(\mathcal{M}_i)_{i\in I}$ is a family of models with elementary maps $\iota_{i,j} : \mathcal{M}_i \to \mathcal{M}_j$ for each $i \leq j$, such that

$$\iota_{i,k} = \iota_{j,k} \circ \iota_{i,j} \text{ for all } i \leq j \leq k$$

then we can construct a *limit*, denoted by $\varinjlim_i \mathcal{M}_i$, as follows: The universe of $\varinjlim_i \mathcal{M}_i$ is

$$\Big(\bigcup_{i\in I} \mathcal{M}_i\Big)\Big/\sim \quad \text{where} \quad \sim := \Big\{(x, y) \,\Big|\, \exists i, j \leq k \in I :\; x \in \mathcal{M}_i, y \in \mathcal{M}_j \wedge \iota_{i,k}(x) = \iota_{j,k}(x)\Big\}.$$

We interpret formulas in a canonical way. Let for example R be a unary relation sybol. Then we interpret R as

$$\big\{[x]_\sim \mid \exists i \in I :\; x \in \mathcal{M}_i \wedge \mathcal{M}_i \models R(x)\big\}.$$

By elementarity of the maps $\iota_{i,j}$, this is well-defined, and $\varinjlim_i \mathcal{M}_i$ is a model of T. Moreover, for each $j \in I$, the map

$$\iota_j : \begin{cases} \mathcal{M}_j \to \varinjlim_i \mathcal{M}_i \\ x \mapsto [x]_\sim \end{cases}$$

is an elementary embedding.
Now, $\varinjlim_i \mathcal{M}_i$ has a universal property like the one described above: The diagram

$$\begin{tikzcd}
\mathcal{M}_j \arrow[rr, "\iota_{j,k}"] \arrow[dr, "\iota_j"'] & & \mathcal{M}_k \arrow[dl, "\iota_k"] \\
& \varinjlim_i \mathcal{M}_i &
\end{tikzcd}$$

commutes, and for any $\mathcal{N} \models T$ with elementary maps $\iota'_i : \mathcal{M}_i \to \mathcal{N}$ that also makes such a diagram commute, there exists a unique elementary map $\varinjlim_i \mathcal{M}_i \to \mathcal{N}$ that makes the following diagram commute.

$$\begin{tikzcd}
\mathcal{M}_j \arrow[rr, "\iota_{j,k}"] \arrow[dr, "\iota_j"'] \arrow[ddr, bend right, "\iota'_j"'] & & \mathcal{M}_j \arrow[dl, "\iota_k"] \arrow[ddl, bend left, "\iota'_k"] \\
& \varinjlim_i \mathcal{M}_i \arrow[d, dashed] & \\
& \mathcal{N} &
\end{tikzcd}$$

One can easily check that the union of an elementary chain of models - as constructed above - is a special case of a directed colimit.

It is a well-known and remarkably useful fact that equivalences of categories preserve all limits and colimits (see Theorem 4.5.2 and Theorem 4.5.3 in Riehl [24]), so in particular they preserve directed colimits. We deduce the following result.

Proposition 4.8. *Let T and T' be theories that admit QE. If F : $\boldsymbol{Mod}(T) \to \boldsymbol{Mod}(T')$ is an equivalence of categories, and $(\mathcal{M}_\lambda)_{\lambda<\kappa}$ is an elementary chain of models of T, then*

$$\varinjlim_{\lambda<\kappa} F(\mathcal{M}_\lambda) \simeq F(\mathcal{M}^*) = \bigcup_{\lambda<\kappa} \mathrm{im}\left[F(\subseteq_\lambda)\right]$$

where $\mathcal{M}^* := \bigcup_{\lambda<\kappa} \mathcal{M}_\lambda$ and $\subseteq_\lambda$ is the inclusion map from $\mathcal{M}_\lambda \to \mathcal{M}^*$.

Proof. Note first that since F preserves directed colimits, we know that $F(\mathcal{M}^*) \simeq \varinjlim F(\mathcal{M}_\lambda)$ w.r.t. the elementary maps $F(\subseteq_{\mu,\lambda}) : F(\mathcal{M}_\mu) \to F(\mathcal{M}_\lambda)$.
Next, note that for each $\lambda < \kappa$, the map $F(\subseteq_\lambda) : F(\mathcal{M}_\lambda) \to F(\mathcal{M}^*)$ is an elementary map, and so $\mathcal{M}'_\lambda := \mathrm{im}\left[F(\subseteq_\lambda)\right]$ is a model of T'. Moreover,

$$\mathcal{M}'_\lambda = \mathrm{im}\left[F(\subseteq_\lambda)\right] = \mathrm{im}\left[F(\subseteq_{\lambda+1} \circ \subseteq_{\lambda,\lambda+1})\right] \subseteq \mathrm{im}\left[F(\subseteq_{\lambda+1})\right] = \mathcal{M}'_{\lambda+1}$$

for all $\lambda < \kappa$, and by QE, any chain of submodels is elementary, so $(\mathcal{M}'_\lambda)_{\lambda<\kappa}$ is an elementary chain of models of T'. Therefore, $\bigcup_\lambda \mathcal{M}'_\lambda$ is a model of T'.
Moreover, it is clear that the diagram below commutes for all $\nu < \mu < \kappa$.

$$\begin{tikzcd}
F(\mathcal{M}_\nu) \arrow[rr, "F(\subseteq)"] \arrow[dr, "F(\subseteq)'"'] & & F(\mathcal{M}_\mu) \arrow[dl, "F(\subseteq)'"] \\
& \bigcup_\lambda \mathrm{im}\left[F(\subseteq_\lambda)\right] &
\end{tikzcd}$$

where $F(\subseteq_\nu)'$ is the corestriction of $F(\subseteq_\nu)$ to $\bigcup_\lambda \mathrm{im}\left[F(\subseteq_\lambda)\right]$. Therefore, by the universal property of $F(\mathcal{M}^*) \simeq \varinjlim F(\mathcal{M}_\lambda)$, there must exist a unique elementary embedding

$$F(\mathcal{M}^*) \to \bigcup_{\lambda<\kappa} \mathrm{im}\left[F(\subseteq_\lambda)\right]$$

that makes the following diagram commute for all $\nu < \mu < \kappa$.

$$\begin{tikzcd}
F(\mathcal{M}_\nu) \arrow[rr, "F(\subseteq)"] \arrow[dr, "F(\subseteq)"'] \arrow[ddr, bend right, "F(\subseteq)'"'] & & F(\mathcal{M}_\mu) \arrow[dl, "F(\subseteq)"] \arrow[ddl, bend left, "F(\subseteq)'"] \\
& F(\mathcal{M}^*) \arrow[d, dashed] & \\
& \bigcup_\lambda \operatorname{im}\big[F(\subseteq_\lambda)\big] &
\end{tikzcd}$$

It is clear that this can only be the inclusion map, so we conclude that

$$F(\mathcal{M}^*) \subseteq \bigcup_{\lambda<\kappa} \operatorname{im}\big[F(\subseteq_\lambda)\big]$$

and the inverse inclusion follows immediately, since $F(\subseteq_\mu) : F(\mathcal{M}_\mu) \to F(\mathcal{M}^*)$ for all $\mu < \kappa$. $\square$

Corollary 4.9. *If T and T' are countable theories that admit QE, then any equivalence $\boldsymbol{Mod}(T) \simeq \boldsymbol{Mod}(T')$ maps countable models of T to countable models of T' and vice-versa. Moreover, if $\boldsymbol{Mod}(T) \simeq \boldsymbol{Mod}(T')$ and T is $\aleph_0$-categorical, then T' is as well.*

Proof. Let $F : \mathbf{Mod}(T) \rightleftarrows \mathbf{Mod}(T') : G$ with natural isomorphisms

$$\eta : FG \Rightarrow \mathrm{id}_{\mathbf{Mod}(T')} \ \text{ and } \epsilon : \mathrm{id}_{\mathbf{Mod}(T)} \Rightarrow GF$$

be an equivalence of categories, and $\mathcal{M} \in \mathbf{Mod}(T)$ any model.

<u>Claim:</u> If $F(\mathcal{M})$ is countable, then $\mathcal{M}$ is countable.
We can write (by iterated application downward Löwenheim-Skolem) $\mathcal{M}$ as the union of a strict elementary chain of countable models $(\mathcal{M}_\lambda)_{\lambda<\kappa}$, and the chain is elementary by QE, so $\mathcal{M} = \bigcup_{\lambda<\kappa} \mathcal{M}_\lambda$ as structures. Then, by Proposition 4.8, we get

$$F(\mathcal{M}) = \bigcup_{\lambda<\kappa} \operatorname{im}\big[F(\subseteq_\lambda)\big]$$

and since $F(\mathcal{M})$ is countable, there exists a countable subsequence $\lambda_0 < \lambda_1 < \dots$ in κ such that

$$F(\mathcal{M}) = \bigcup_{n\in\omega} \operatorname{im}\big[F(\subseteq_{\lambda_n})\big].$$

Now, applying G to both sides, and again using Proposition 4.8, we get

$$\mathcal{M} \overset{\epsilon}{\simeq} GF(\mathcal{M}) \overset{4.8}{\simeq} \bigcup_{n\in\omega} \operatorname{im}[GF(\subseteq_{\lambda_n})] \overset{\epsilon}{\simeq} \bigcup_{n\in\omega} \mathcal{M}_{\lambda_n}.$$

Since the models $\mathcal{M}_{\lambda_n}$ are all countable, we conclude that $\mathcal{M}$ is countable.
The proof that G preserves countability is analogous.

Assume now that T is $\aleph_0$-categorcal and let $\mathcal{M}'_1$ and $\mathcal{M}'_2$ be countable models of T'. By essential surjectivity of F, we can find $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{Mod}(T)$ such that $\mathcal{M}'_i \simeq F(\mathcal{M}_i)$ for $i = 1, 2$. By the claim we just proved, $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable, and since T is $\aleph_0$-categorical, they are isomorphic. Hence, we conclude that

$$\mathcal{M}'_1 \simeq F(\mathcal{M}_1) \simeq F(\mathcal{M}_2) \simeq \mathcal{M}'_2.$$

□

For a theory T, we shall denote by $\mathbf{Mod}_\omega(T)$ the subcategory of $\mathbf{Mod}(T)$ containing only the countable models of T.

Remark 4.10. *If T is $\aleph_0$-categorical, then the inclusion functor from $\boldsymbol{End}(\mathcal{M}_T)$ to $\boldsymbol{Mod}_\omega(T)$ is essentially surjective (and clearly fully faithful), so that $\boldsymbol{End}(\mathcal{M}_T) \simeq \boldsymbol{Mod}_\omega(T)$.*

A similar argument as that for Corollary 4.9 shows that the converse of Proposition 4.5 holds, i.e. that $\mathbf{End}(\mathcal{M}_T)$ uniquely determines $\mathbf{Mod}(T)$ if T is $\aleph_0$-categorical.

Theorem 4.11. *If T, T' are $\aleph_0$-categorical and $\boldsymbol{End}(\mathcal{M}_T) \simeq \boldsymbol{End}(\mathcal{M}_{T'})$, then $\boldsymbol{Mod}(T) \simeq \boldsymbol{Mod}(T')$.*

Proof. Let $F : \mathbf{End}(\mathcal{M}_T) \to \mathbf{End}(\mathcal{M}_{T'})$ be an equivalence of categories. By the above remark, we can extend this to an equivalence $F : \mathbf{Mod}_\omega(T) \simeq \mathbf{Mod}_\omega(T')$.
We will define from this a functor $G : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$. Firstly, we define G as F on $\mathbf{Mod}_\omega(T)$.
Fix any uncountable model $\mathcal{M}$ of T. Let $(\mathcal{M}_i)_{i \in I}$ be the family of all countable elementary submodels of $\mathcal{M}$. Then I is a directed set, under the partial order

$$i \leq j \iff \mathcal{M}_i \preccurlyeq \mathcal{M}_j,$$

since - by downward Löwenheim-Skolem - for any countable $\mathcal{M}_i, \mathcal{M}_j \preccurlyeq \mathcal{M}$, we can find $\mathcal{M}_k \succcurlyeq \mathcal{M}_i, \mathcal{M}_j$. Moreover, we can write - again by downward Löwenheim-Skolem - $\mathcal{M}$ as the union of all its countable submodels, that is,

$$\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i = \varinjlim_i \mathcal{M}_i.$$

Denote the inclusion map $\mathcal{M}_i \to \mathcal{M}$ by $\subseteq_i$. We define

$$G(\mathcal{M}) := \varinjlim_{i\in I} F(\mathcal{M}_i) \in \mathbf{Mod}(T').$$

Now, for another uncountable model $\mathcal{N} \models T$, we can write, in the same way, $\mathcal{N} = \varinjlim_{j\in J} \mathcal{N}_j$ where $\mathcal{N}_j$ are the countable elementary submodels of $\mathcal{N}$. For $f : \mathcal{M} \to \mathcal{N}$ an elementary map, we need to define

$$G(f) : \varinjlim_{i\in I} F(\mathcal{M}_i) \to \varinjlim_{j\in J} F(\mathcal{N}_j).$$

Let $x \in F(\mathcal{M}_i)$, and find $j \in J$ such that $f(\mathcal{M}_i) \subseteq \mathcal{N}_j$. Such an index $j \in J$ exists, since $\mathcal{N}$ is uncountable, and $f(\mathcal{M}_i)$ is countable.
Then, define

$$G(f)([x]_\sim) := \Big[F\big(f|_i^j\big)(x)\Big]_\sim$$

where $f\big|_i^j$ is the corestriction of $f|_{\mathcal{M}_i}$ to $\mathcal{N}_j$.
To see that this is well-defined, let $y = F(\subseteq_{i,i'})(x)$ for some $\mathcal{M}_{i'} \succcurlyeq \mathcal{M}_i$, and $j' \in J$ be such that $f(\mathcal{M}_{i'}) \subseteq \mathcal{M}_{j'}$, and assume w.l.o.g. that $\mathcal{N}_{j'} \succ \mathcal{N}_j$. Then

$$F\big(f|_{i'}^{j'}\big)(y) = F\big(f|_{i'}^{j'} \circ \subseteq_{i,i'}\big)(x) = F(\subseteq_{j,j'} \circ f|_i^j)(x),$$

and thus

$$F\big(f|_{i'}^{j'}\big)(y) \sim F\big(f|_i^j\big)(x).$$

Finally, for a countable model $\mathcal{M}'$ and an elementary map $f : \mathcal{M}' \to \mathcal{M}$, we get $f(\mathcal{M}') = \mathcal{M}_i$ for some $i \in I$, so we define $G(f) := \iota_i \circ F(f|^i)$ where ι_i is the limit map $F(\mathcal{M}_i) \to \varinjlim_i F(\mathcal{M}_i)$.
It is easy to check that G forms a functor from $\mathbf{Mod}(T)$ to $\mathbf{Mod}(T')$.

<u>Claim:</u> G is essentially surjective.
This follows from the fact that each model of T' can be written as a directed colimit of its countable models, as described above, and F is essentially surjective.

G is fully faithful on countable models, since the (co)restriction $G : \mathbf{Mod}_\omega(T) \to \mathbf{Mod}_\omega(T')$ is exactly F, which is an equivalence by assumption. Moreover, for an uncountable model $\mathcal{M} = \varinjlim_i \mathcal{M}_i$ and a countable model $\mathcal{M}' \models T$,

$$G : \operatorname{Hom}\big(\mathcal{M}', \varinjlim_i \mathcal{M}_i\big) \to \operatorname{Hom}\big(F(\mathcal{M}'), \varinjlim_i F(\mathcal{M}_i)\big)$$

as defined above, is bijective, since F is f.f. and any elementary map $F(\mathcal{M}') \to \varinjlim_i F(\mathcal{M}_i)$ factors through some ι_i.

To finish the proof, we fix two uncountable models, $\mathcal{M} = \varinjlim_i \mathcal{M}_i$ and $\mathcal{N} = \varinjlim_j \mathcal{N}_j$, as above, and show that

$$G : \mathrm{Hom}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}(G(\mathcal{M}), G(\mathcal{N}))$$

is bijective.

<u>Claim:</u> $G : \mathrm{Hom}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}(G(\mathcal{M}), G(\mathcal{N}))$ is surjective.

Let $g : \varinjlim_\lambda \mathcal{M}_i \to \varinjlim_j \mathcal{N}_j$ be an elementary map. For each $i \in I$, we can find (since $\mathcal{N}$ is uncountable) $j(i) \in J$ such that

$$\forall x \in F(\mathcal{M}_i) \exists y(x) \in F(\mathcal{N}_{j(i)}) : \; g([x]_\sim) = [y]_\sim.$$

Clearly, such $y(x)$ is unique by injectivity of the elementary maps $F(\subseteq)$. Now, it is not hard to see that

$$g_i : \begin{cases} F(\mathcal{M}_i) \to F(\mathcal{N}_{j(i)}) \\ x \mapsto y(x) \end{cases}$$

is an elementary map, so by fullness of F, there exists $f_i : \mathcal{M}_i \to \mathcal{N}_{j(i)}$ such that $g_i = F(f_i)$. One can check that

$$g = G\big(\bigcup_{i \in I} f_i\big).$$

<u>Claim:</u> $G : \mathrm{Hom}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}(G(\mathcal{M}), G(\mathcal{N}))$ is injective.

Let $f_1, f_2 : \mathcal{M} \to \mathcal{N}$ be distinct elementary maps. Find $i \in I$ such that f_1 and f_2 do not agree on $\mathcal{M}_i$, and $j \in J$ such that $f_1(\mathcal{M}_i) \cup f_2(\mathcal{M}_i) \subseteq \mathcal{N}_j$. Then $f_1(\mathcal{M}_i), f_2(\mathcal{M}_i) \preccurlyeq \mathcal{N}_j$ by QE, and by faithfulness of G on countable models,

$$G((f_1)|_i^j) \neq G((f_2)|_i^j),$$

so it follows by definition of G on morphisms, that $G(f_1) \neq G(f_2)$. $\square$

Chapter 5

Interpreting one Structure in Another

One of the fundamental tasks in many fields of modern mathematics, including model theory, is to characterize the studied objects, by identifying objects that are 'as good as equal' - or *isomorphic*. But before one can even begin to ask whether two objects are isomorphic, one needs to find a suitable notion of an *isomorphism*. Whether or not two objects are 'as good as equal' depends heavily on the context. From a purely set-theoretic perspective, $\mathbb{N}$ and $\mathbb{Q}$ are the same: Both are countably infinite. However, when we look at their algebraic structure, the two are drastically different. While category theory provides a general framework for comparing objects up to isomorphism, when it comes to working in a specific area of mathematics - a specific category - one needs to agree on a concrete notion of isomorphism in that area.
Model Theory is no exception to this. Intuitively, to a model theorist, the theory $\mathrm{Th}(\langle \mathbb{N}, < \rangle)$ is as good as the theory $\mathrm{Th}(\langle \mathbb{N}, <, 0 \rangle)$, and vice-versa. This is because we can already identify 0 in $\langle \mathbb{N}, < \rangle$ (via the formula $\varphi(u) := \forall v \; u < v$). Similarly, the theory $\mathrm{Th}(\langle \mathbb{N}, \leq \rangle)$ is as good as either of these theories: We can define $<$ in terms of $\leq$ and vice-versa. So what is it that makes all these theories 'the same'?
As it turns out, the appropriate notion of 'sameness' to use in the context of theories is that of *bi-interpretability*. There have historically been several, incompatible definitions of what an *interpretation* of a theory in another is. Makkai and Reyes [19] make the notion precise, in the setting of "first-order categorical logic", where theories correspond to certain categories (called *logical categories*), and an interpretation of a theory in another is a functor with special properties (called *logical functor*) between these categories.
We are concerned here only with the $\aleph_0$-categorical case, where an interpretation of a theory in another is simply an interpretation of a countable model in another.

We follow the definitions of Ahlbrandt and Ziegler [2], which serves as a basis for this chapter. Ahlbrandt and Ziegler also give a proof in that paper of one of the fundamental reconstruction theorems of model theory - namely that $\aleph_0$-categorical theories are determined up to bi-interpretability by the automorphism group of their unique countable model, equipped with the topology of pointwise convergence. For historical reasons stated in the introduction, we refer to this as the MCAZ Theorem. We present a proof of it in Section 5.2.

5.1 Bi-Interpretability

Let $\mathcal{L}$ and $\mathcal{K}$ denote languages, and let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\mathcal{N}$ a $\mathcal{K}$-structure.

Definition 5.1. An *interpretation* of $\mathcal{N}$ in $\mathcal{M}$, denoted by $I : \mathcal{M} \rightsquigarrow \mathcal{N}$, is a tuple $I = (U, i)$ where

- $U = \varphi(\mathcal{M}^n) \subseteq \mathcal{M}^n$ is a definable set, for some $n \in \omega$, and
- $i : U \to \mathcal{N}$ is a surjection,

such that for any (m-ary) relation $P \in \mathcal{K}$, (m-ary) function $f \in \mathcal{K}$, and constant $c \in \mathcal{K}$, the following sets are definable in $\mathcal{M}$.

(i) $\equiv^i := \{(\overline{x}, \overline{y}) \in U^2 \mid i(\overline{x}) = i(\overline{y})\}$.
(ii) $P^i := \{(\overline{x}_1, \dots, \overline{x}_m) \in U^m \mid \big(i(\overline{x}_1), \dots, i(\overline{x}_m)\big) \in P^{\mathcal{N}}\}$.
(iii) $f^i := \{(\overline{x}_1, \dots, \overline{x}_m, \overline{y}) \in U^{m+1} \mid f^{\mathcal{N}}(i(\overline{x}_1), \dots, i(\overline{x}_m)) = i(\overline{y})\}$.
(iv) $c^i := \{\overline{x} \in U \mid i(\overline{x}) = c^{\mathcal{N}}\}$.

If an interpretation $I : \mathcal{M} \rightsquigarrow \mathcal{N}$ exists, we say that $\mathcal{N}$ is *interpretable* in $\mathcal{M}$.

Remark 5.2. *Given isomorphisms $f : \mathcal{M} \to \mathcal{M}'$ and $g : \mathcal{N} \to \mathcal{N}'$, any interpretation $(U, i) : \mathcal{M} \rightsquigarrow \mathcal{N}$ induces an interpretation $\big(f(U), g \circ i \circ f^{-1}\big) : \mathcal{M}' \rightsquigarrow \mathcal{N}'$. In particular, isomorphisms preserve interpretability.*

By the above remark, it is sensible to define an interpretation of an $\aleph_0$-categorical theory in another, as an interpretation of one countable model in another.

Definition 5.3. Let T be an $\aleph_0$-categorical $\mathcal{L}$-theory, S an $\aleph_0$-categorical $\mathcal{K}$-theory, and $\mathcal{M} \models T$, $\mathcal{N} \models T$ countable models. For an interpretation $I : \mathcal{M} \rightsquigarrow \mathcal{N}$, we say that I is an interpretation of S in T and write $I : T \rightsquigarrow S$.

It is easy to see that if $(U,i) : \mathcal{M} \rightsquigarrow \mathcal{N}$ is an interpretation, then $U/\equiv^i$ forms a $\mathcal{K}$-structure, and the canonical map $i : U/\equiv^i \to \mathcal{N}$ is an isomorphism.

For the sake of readability, the definition of an interpretation given above is only one-sorted structures. We can generalize it in a canonical manner to that of an interpretation of a many-sorted structure $\mathcal{N}$ in another many-sorted structure $\mathcal{M}$: For each sort s in $\mathcal{N}$, there is a definable set U_s of tuples (possibly of mixed sorts) in $\mathcal{M}$, and a surjection $i_s : U_s \to \mathcal{N}_s$ such that the relation $i_s(\overline{x}) = i_s(\overline{y})$ is definable in $\mathcal{M}$. Relations, functions, and constants are handled as one would expect.

Example 5.4. Consider the structures $\langle \mathbb{Q}, +, \cdot, 0, 1\rangle$ and $\langle \mathbb{Z}, +, \cdot, 0, 1\rangle$. Clearly, these structures have very different theories. However, we can interpret $\mathbb{Z}$ in $\mathbb{Q}$, since $\mathbb{Z}$ is a definable subset of $\mathbb{Q}$ (see for example Robinson [25]). Moreover, we can interpret $\mathbb{Q}$ in $\mathbb{Z}$: Let

$$U := \{(p,q) \in \mathbb{Z}^2 \mid q \neq 0\}$$

and

$$i : \begin{cases} U \to \mathbb{Q} \\ (p,q) \mapsto \frac{p}{q}. \end{cases}$$

One can check that this is indeed an interpretation $\mathbb{Z} \rightsquigarrow \mathbb{Q}$. For example, we get

$$\equiv^i \quad = \quad \Big\{ \big((p,q),(p',q')\big) \in U^2 \mid p \cdot q' = p' \cdot q \Big\}.$$

In order to define formally the notion of a bi-interpretation, we describe here how interpretations are composed, define identity-interpretations $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \rightsquigarrow \mathcal{M}$, and describe how each interpretation $I : \mathcal{M} \rightsquigarrow \mathcal{M}'$ induces a map $I : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$. This hints at the fact that interpretations are the functors between suitably defined categories of theories [19].

Suppose $I = (U,i) : \mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ and $J = (V,j) : \mathcal{M}_2 \rightsquigarrow \mathcal{M}_3$ are both interpretations, where $U \subseteq \mathcal{M}_1^n$ and $V \subseteq \mathcal{M}_2^m$. Then we define the composite interpretation

$$J \circ I = (W,k) : \mathcal{M}_1 \rightsquigarrow \mathcal{M}_3$$

as follows. Let

$$W := i^{-1}(V) = \{(\overline{x}_1, \ldots, \overline{x}_m) \in U^m \mid (i(\overline{x}_1), \ldots, i(\overline{x}_m)) \in V\}.$$

This is definable, sine U is definable in $\mathcal{M}_1$, and V is definable in $\mathcal{M}_2$, and therefore its pullback is definable in $\mathcal{M}_1$. Moreover, let

$$k : \begin{cases} W \to \mathcal{M}_3 \\ (\overline{x}_1, \ldots, \overline{x}_m) \mapsto j(i(\overline{x}_1), \ldots, i(\overline{x}_m)). \end{cases}$$

It is not difficult to check that (W, k) is indeed an interpretation of $\mathcal{M}_3$ in $\mathcal{M}_1$.

Note that for any structure $\mathcal{M}$, the identity $\mathrm{id}_{\mathcal{M}} := (\mathcal{M}, id_{\mathcal{M}})$ is clearly an interpretation $\mathcal{M} \rightsquigarrow \mathcal{M}$, and this forms an identity under composition, i.e. $I \circ \mathrm{id}_{\mathcal{M}} = I$ for each $I : \mathcal{M} \rightsquigarrow \mathcal{N}$ and $\mathrm{id}_{\mathcal{M}} \circ J = J$ for each $J : \mathcal{N} \rightsquigarrow \mathcal{M}$.
Finally, note that if $I = (U, I) : \mathcal{M} \rightsquigarrow \mathcal{N}$ is an interpretation and $f : \mathcal{M} \to \mathcal{M}$ is an elementary map, then there exists a unique elementary map $I(f) : \mathcal{N} \to \mathcal{M}$ that makes the following diagram commute, where $f : U \to U$ is defined pointwise (note that $f(U) \subseteq U$ since U is definable).

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i"'] & U \arrow[d, "i"] \\ \mathcal{N} \arrow[r, "I(f)"'] & \mathcal{M} \end{tikzcd} \tag{5.1}$$

Concretely, $I(f)$ maps each element $i(\overline{x})$ of $\mathcal{N}$ to $i(f(\overline{x}))$. This is well-defined, since $\equiv^i$ is definable, and therefore preserved by f, i.e. $i(\overline{x}) = i(\overline{y}) \implies i(f(\overline{x})) = i(f(\overline{y}))$.

Definition 5.5. Given interpretations $I_1 = (U_1, i_1), I_2 = (U_2, i_2) : \mathcal{M} \rightsquigarrow \mathcal{N}$, we say that I_1 and I_2 are *homotopic* (write $I_1 \sim I_2$) if the set

$$\{(x, y) \in U_1 \times U_2 \mid i_1(x) = i_2(y)\}$$

is definable in $\mathcal{M}$.

Definition 5.6. We say that $\mathcal{M}$ and $\mathcal{N}$ are *mutually interpretable* if there exist interpretations $I : \mathcal{M} \rightsquigarrow \mathcal{N}$ and $J : \mathcal{N} \rightsquigarrow \mathcal{M}$.
If, moreover, $J \circ I \sim \mathrm{id}_{\mathcal{M}}$ and $I \circ J \sim \mathrm{id}_{\mathcal{N}}$, then we call (I, J) a *bi-interpretation* and say that $\mathcal{M}$ and $\mathcal{N}$ are *bi-interpretable*.

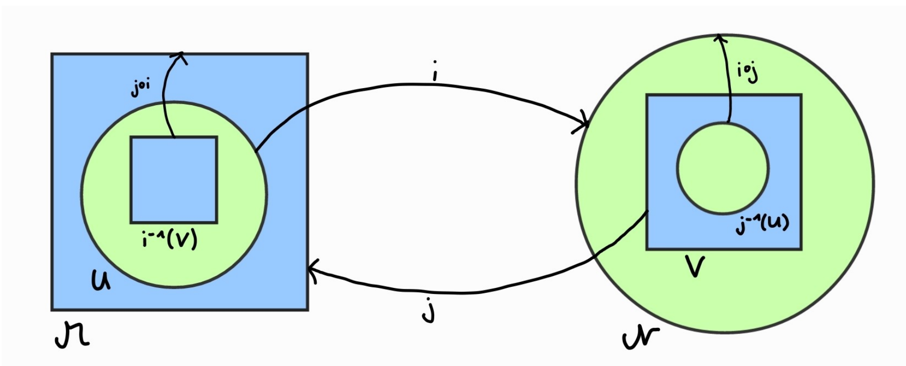


Figure 5.1: Bi-Interpretation between $\mathcal{M}$ and $\mathcal{N}$, adopted from [7]

Remark 5.7. *Bi-interpretability forms an equivalence relation. Reflexivity holds since* $\mathrm{id}_{\mathcal{M}} \circ \mathrm{id}_{\mathcal{M}} \sim \mathrm{id}_{\mathcal{M}}$. *For transitivity, let* $I : \mathcal{M}_1 \leftrightsquigarrow \mathcal{M}_2 : J$, *and* $I' : \mathcal{M}_2 \leftrightsquigarrow \mathcal{M}_3 : J'$ *be bi-interpretations. Then*

$$(J \circ J') \circ (I' \circ I) \sim J \circ \mathrm{id}_{\mathcal{M}_2} \circ I \sim J \circ I \sim \mathrm{id}_{\mathcal{M}},$$

and similarly, $(I' \circ I) \circ (J \circ J') \sim \mathrm{id}_{\mathcal{M}_3}$, *so* $I' \circ I : \mathcal{M}_1 \leftrightsquigarrow \mathcal{M}_3 : J \circ J'$ *is a bi-interpretation. Symmetry is clear.*

One can define in general what it means for two theories, T and T', to be bi-interpretable. Loosely speaking, an interpretation of T' in T should be a 'recipe' for interpreting any model of T' in any model of T. Since we are only interested in bi-interpretability in the context of results about reconstructibility of $\aleph_0$-categorical theories up to bi-interpretation, we give here a concrete definition of bi-interpretability between theories, only in the $\aleph_0$-categorical case.

Definition 5.8. For $\aleph_0$-categorcial theories T, T', we say that T and T' are bi-interpretable (write $T \simeq T'$) if their respective countable models are bi-interpretable.

Note that if $\mathcal{M}$ and $\mathcal{M}'$ are models of the same theory, and $f : \mathcal{M} \to \mathcal{M}'$ is an isomorphism, then $(\mathcal{M}, f) : \mathcal{M} \leftrightsquigarrow \mathcal{M} : (\mathcal{M}', f^{-1})$ forms a bi-interpretation. Therefore, the definition above is sensible.

Proposition 5.9. $\mathcal{M}$ *and* $\mathcal{M}^{eq}$ *are bi-interpretable.*

Proof. Firstly, define $J = (V, j) : \mathcal{M}^{eq} \rightsquigarrow \mathcal{M}$ in the obvious way, namely $V := S_=$ and

$$j : \begin{cases} V \to \mathcal{M} \\ [x]_= \mapsto x. \end{cases}$$

Now, define $I = \big((U_E, i_E)\big)_E : \mathcal{M} \rightsquigarrow \mathcal{M}^{eq}$ as follows. Firstly, let $U_= := S_=$, and let $i_=$ map each $[x]_=$ to x. Now, for any definable equivalence relation $E = \varphi(\mathcal{M}^n \times \mathcal{M}^n)$, define

$$U_E := \mathcal{M}^n$$

and

$$i_E : \begin{cases} U_E \to S_E \\ \overline{x} \mapsto [\overline{x}]_E. \end{cases}$$

It is easy to check that both I and J are interpretations, and $J \circ I \sim \mathrm{id}_{\mathcal{M}}$, and $I \circ J \sim \mathrm{id}_{\mathcal{M}^{eq}}$. □

Bi-interpretations preserve most of the properties of a structure/theory that we know. For example, the automorphism groups of two bi-interpretable structures are isomorphic as topological groups.

Lemma 5.10. *For a bi-interpretation $I = (U, i) : \mathcal{M} \leftrightsquigarrow \mathcal{M}' : (V, j) = J$, the induced maps*

$$I : \mathrm{Aut}(\mathcal{M}) \rightleftarrows \mathrm{Aut}(\mathcal{M}') : J$$

are continuous group-homomorphisms with $J \circ I = \mathrm{id}$ and $I \circ J = \mathrm{id}$.

Proof. Recall that I induces a map $I : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$, as in 5.1, and similarly J induces a map $J : \mathrm{Aut}(\mathcal{M}') \to \mathrm{Aut}(\mathcal{M})$.

<u>Claim:</u> $I : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$ is homomorphic.
For $f, g \in \mathrm{Aut}(\mathcal{M})$, the two inner squares in the following diagram commute by 5.1, and so the outer square commutes as well.

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "i"'] & U \arrow[r, "g"] \arrow[d, "i"] & U \arrow[d, "i"] \\
\mathcal{M}' \arrow[r, "I(f)"'] & \mathcal{M}' \arrow[r, "I(g)"'] & \mathcal{M}'
\end{tikzcd}$$

Therefore,

$$I(g \circ f) = I(g) \circ I(f).$$

<u>Claim:</u> $I : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$ is continuous.
We fix some $y \in \mathcal{M}'$, and show that $I^{-1}(\mathrm{Aut}_y(\mathcal{M}'))$ is open in $\mathrm{Aut}(\mathcal{M})$. Find and fix by surjectivity of i some $x \in i^{-1}(y)$. Then for $f \in \mathrm{Aut}(\mathcal{M})$,

$$f \in I^{-1}\big(\mathrm{Aut}_y(\mathcal{M}')\big) \iff I(f)(i(x)) = y \overset{5.1}{\iff} (i \circ f)(x) = y \iff f(x) \in i^{-1}(y),$$

so $I^{-1}(\mathrm{Aut}_y(\mathcal{M}'))$ is open in $\mathrm{Aut}(\mathcal{M})$.

We conclude that $I : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$, and analogously, $J : \mathrm{Aut}(\mathcal{M}') \to \mathrm{Aut}(\mathcal{M})$, are continuous group-homomorphisms.

<u>Claim:</u> $(J \circ I)(f) = f$ for all $f \in \mathrm{Aut}(\mathcal{M})$.
Fix $f \in \mathrm{Aut}(\mathcal{M})$. To show that $J\big(I(f)\big) = f$, we just need to check that the following diagram commutes.

$$\begin{tikzcd}
V \arrow[r, "I(f)"] \arrow[d, "j"'] & V \arrow[d, "j"] \\
\mathcal{M} \arrow[r, "f"'] & \mathcal{M}
\end{tikzcd}$$

Let $\overline{y} \in V$. Find $\overline{x}_1, \ldots, \overline{x}_m \in U$ such that $\big(i(\overline{x}_1), \ldots, i(\overline{x}_m)\big) = \overline{y}$. Since $J \circ I \sim \mathrm{id}_{\mathcal{M}}$, we know that the set

$$\Big\{\big(\overline{x}_1, \ldots, \overline{x}_m, x\big) \in i^{-1}(V) \times \mathcal{M} \mid j\big(i(\overline{x}_1), \ldots, i(\overline{x}_m)\big) = x\Big\} \tag{5.2}$$

is definable, and therefore, invariant under f. So, we get

$$\begin{aligned} \big[j \circ I(f)\big](\overline{y}) &= \big[j \circ I(f)\big]\big(i(\overline{x}_1), \ldots, i(\overline{x}_m)\big) \\ &\overset{5.1}{=} j\big(i(f\overline{x}_1), \ldots, i(f\overline{x}_m)\big) \\ &= f\Big[j\big(i(\overline{x}_1), \ldots, i(\overline{x}_m)\big)\Big] = \big[f \circ j\big](\overline{y}). \end{aligned}$$

Thus, we have shown that the desired diagram commutes, and therefore that $J \circ I = \mathrm{id}_{\mathrm{Aut}(\mathcal{M})}$. We obtain $I \circ J = \mathrm{id}_{\mathrm{Aut}(\mathcal{M}')}$ analogously. □

5.2 The Makkai-Coquand-Ahlbrandt-Ziegler Theorem

Lemma 5.10 tells us that we can reconstruct the topological automorphism group of a structure up to isomorphism, given its theory up to bi-interpretability. For structures/theories which are $\aleph_0$-categorical, the inverse is true as well. This result is attributed by Ahlbrandt and Ziegler [2] to unpublished notes by Coquand. It will serve as a central tool in our proof of Lascar's Reconstruction Theorem 9.16. We present in this section a proof of the Makkai-Coquand-Ahlbrandt-Ziegler Theorem, which is adapted from Ahlbrandt and Ziegler [2, Corollary 1.4].

Theorem 5.11 (MCAZ)**.** *Two $\aleph_0$-categorical theories, T and T', are bi-interpretable if and only if* $\mathrm{Aut}(\mathcal{M}_T)$ *and* $\mathrm{Aut}(\mathcal{M}_{T'})$ *are isomorphic as topological groups.*

We begin by proving the following.

Theorem 5.12 ([2, Theorem 1.21])**.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be $\aleph_0$-categorical structures, and $\gamma : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$ a continuous homomorphism. Then there exists an interpretation $I : \mathcal{M} \rightsquigarrow \mathcal{M}'$ such that $\gamma(f) = I(f)$ for all $f \in \mathrm{Aut}(\mathcal{M})$, if and only $G := \mathrm{im}(\gamma)$ acts oligomorphically on $\mathcal{M}'$.*

Proof. Let $\mathcal{M}, \mathcal{M}'$ and γ be given as in the statement.
Assume first that γ has the form $f \mapsto I(f)$ for some interpretation $I = (U, i) : \mathcal{M} \rightsquigarrow \mathcal{M}'$, where $U \subseteq \mathcal{M}^n$. Since $\mathcal{M}$ is $\aleph_0$-categorical, $\mathrm{Aut}(\mathcal{M})$ acts oligomorphically on $\mathcal{M}$ by Ryll-Nardzewski. Therefore, G acts oligomorphically on $\mathcal{M}'$: Suppose it does

not. Then w.l.o.g. there exist $x_1, x_2, \ldots \in U$ such that $i(x_1), i(x_2), \ldots$ are in distinct G-orbits. Since $\mathrm{Aut}(\mathcal{M})$ acts oligomorphically, we know that for some $\ell \neq k$, there exists $f \in \mathrm{Aut}(\mathcal{M})$ with $f(x_\ell) = x_k$. Then, by commutativity of 5.1, we get

$$I(f)(i(x_\ell)) = i(f(x_\ell)) = i(x_k),$$

a contradiction to our assumption.

Assume now that $\gamma : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$ is a continuous homomorphism such that $G = \mathrm{im}(\gamma)$ acts oligomorphically on $\mathcal{M}'$. Then, let $y_1, \ldots, y_k \in \mathcal{M}'$ be representatives of the classes in $\mathcal{M}'/G$, i.e.

$$\mathcal{M}' = \bigsqcup_{r=1}^{k} G \cdot y_r \ . \tag{5.3}$$

Since γ is a continuous homomorphism, preimages of clopen subgroups under γ are clopen subgroups. Therefore, for any $r = 1, \ldots, k$, there exists a tuples $\overline{x}_r \in \mathcal{M}^{<\omega}$ such that

$$\gamma^{-1}\big(\mathrm{Aut}_{y_r}(\mathcal{M}')\big) = \mathrm{Aut}_{\overline{x}_r}(\mathcal{M}) =: H_r. \tag{5.4}$$

We may assume that the tuples $\overline{x_r}$ all have the same length $n \in \omega$, since otherwise we can simply pad the tuples by repeating elements. Moreover, we can assume that all $\overline{x}_r$ are pairwise distinct, since we may pad them distinctly otherwise.
Let $\overline{X} := \overline{x}_1 \ldots \overline{x}_r$ be the concatenation of these tuples. Then by (5.4), we get

$$\mathrm{Aut}_{\overline{y}}(\mathcal{M}') = \gamma(\mathrm{Aut}_{\overline{X}}(\mathcal{M})). \tag{5.5}$$

We will now define an interpretation $I = (U, i) : \mathcal{M} \rightsquigarrow \mathcal{M}'$, and prove that for each $f \in \mathrm{Aut}(\mathcal{M})$, $\gamma(f)$ makes the following diagram commute.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "i"'] & U \arrow[d, "i"] \\ \mathcal{M}' \arrow[r, "\gamma(f)"'] & \mathcal{M}' \end{tikzcd} \tag{5.6}$$

Then, by the observations in Section 5.1, we are done.
We define the universe U of the interpretation as

$$U := \big\{ f(\overline{x}_r, \overline{X}) \mid r \leq k \text{ and } f \in \mathrm{Aut}(\mathcal{M}) \big\}.$$

which is definable, since $\mathcal{M}$ is $\aleph_0$-categorical (so types are isolated by Ryll-Nardzewski), and

$$U = \Big\{ \overline{z} \in \mathcal{M}^{(k+1)n} \ \Big| \ \mathrm{tp}(\overline{z}) = \mathrm{tp}(\overline{x}_r, \overline{X}) \vee \cdots \vee \mathrm{tp}(\overline{z}) = \mathrm{tp}(\overline{x}_k, \overline{X}) \Big\}.$$

Moreover, we define the map $i : U \to \mathcal{M}'$ as

$$i\big(f(\overline{x}_r, \overline{X})\big) := \gamma(f)(y_r).$$

This is well-defined, by (5.5). Moreover, i is surjective by (5.3).
Now, consider the pullback $\equiv^i$ of equality. We have

$$\begin{aligned}
\equiv^i &= \Big\{ \big(f(\overline{x}_r, \overline{X}), g(\overline{x}_s, \overline{X})\big) \in U^2 \;\Big|\; 1 \leq r, s \leq k \text{ and } \gamma(f)(y_r) = \gamma(g)(y_s) \Big\} \\
&\overset{5.3}{=} \Big\{ \big(f(\overline{x}_r, \overline{X}), g(\overline{x}_r, \overline{X})\big) \in U^2 \;\Big|\; 1 \leq r \leq k \text{ and } \gamma(f)(y_r) = \gamma(g)(y_r) \Big\} \\
&\overset{5.4}{=} \Big\{ \big(f(\overline{x}_r, \overline{X}), g(\overline{x}_r, \overline{X})\big) \in U^2 \;\Big|\; 1 \leq r \leq k \text{ and } f(\overline{x}_r) = g(\overline{x}_r) \Big\} \\
&= \bigcup_{k=1}^{r} \Big\{ \big((\overline{x}, \overline{z}), (\overline{x}', \overline{z}')\big) \in U^2 \;\Big|\; \overline{x} = \big(z_{(k-1)n+1}, \ldots, z_{kn}\big) = \big(z'_{(k-1)n+1}, \ldots, z'_{kn}\big) = \overline{x}' \Big\}
\end{aligned}$$

which is clearly definable.
Now, let R be a unary relation in the language of $\mathcal{M}'$. Moreover, let $r_1 < \cdots < r_\ell \leq k$ be those indices where $\mathcal{M}' \models R(y_{r_s})$. Then

$$\begin{aligned}
R^i &= \Big\{ f(\overline{x}_r, \overline{X}) \in U \;\Big|\; r \leq k \text{ and } \mathcal{M}' \models R\big(\gamma(f)(y_r)\big) \Big\} \\
&= \Big\{ f(\overline{x}_r, \overline{X}) \in U \;\Big|\; r \leq k \text{ and } \mathcal{M}' \models R(y_r) \Big\} \\
&= \bigcup_{s=1}^{\ell} \big\{ f(\overline{x}_{r_s}, \overline{X}) \in U \big\} = \bigcup_{s=1}^{\ell} \Big\{ (\overline{x}, \overline{z}) \in U \;\Big|\; \overline{x} = \big(z_{(r_s-1)n+1}, \ldots, z_{r_s n}\big) \Big\}
\end{aligned}$$

is definable in $\mathcal{M}$. One can prove similarly that pullbacks of relations of higher arity, functions, and constants are definable in $\mathcal{M}$. We conclude that (U, i) is an interpretation $\mathcal{M} \rightsquigarrow \mathcal{M}'$.
Finally, notice that we have defined i precisely in such a way that for any $f \in \mathrm{Aut}(\mathcal{M})$,

$$\forall \overline{z} \in U : \; i(f(\overline{z})) = \gamma(f)(i(\overline{z})),$$

i.e. diagram (5.6) commutes, and we are done. □

We are now able to prove the MCAZ Theorem 5.11.

Proof of Theorem 5.11. Write $\mathcal{M} := \mathcal{M}_T$ and $\mathcal{M}' := \mathcal{M}_{T'}$.

Suppose first that T and T' are bi-interpretable. Then Lemma 5.10 tells us precisely that $\mathrm{Aut}(\mathcal{M}) \simeq \mathrm{Aut}(\mathcal{M}')$ as topological groups.

Suppose now that $\gamma : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$ is a homeomorphic group-isomorphism. Since T' is $\aleph_0$-categorcial, $\mathrm{im}(\gamma) = \mathrm{Aut}(\mathcal{M}')$ acts oligomorphically on $\mathcal{M}'$. Thus, by

Theorem 5.12, there exists an interpretation $I = (U, i) : \mathcal{M} \rightsquigarrow \mathcal{M}'$ such that $\gamma = I$ on $\mathrm{Aut}(\mathcal{M})$. Setting $\delta := \gamma^{-1}$, we find, by the same argument, an interpretation $J = (V, j) : \mathcal{M}' \rightsquigarrow \mathcal{M}$ with $\delta = J$ on $\mathrm{Aut}(\mathcal{M}')$. Let $n, m \in \omega$ be such that $U \subseteq \mathcal{M}^n$ and $V \subseteq \mathcal{M}'^m$.

<u>Claim:</u> $J \circ I \sim \mathrm{id}_{\mathcal{M}}$.
We need to show that

$$\Big\{\big(\overline{x}_1, \dots, \overline{x}_m, x\big) \in i^{-1}(V) \times \mathcal{M} \;\big|\; j\big(i(\overline{x}_1), \dots, i(\overline{x}_m)\big) = x\Big\}$$

is definable in $\mathcal{M}$. By $\aleph_0$-categoricity of T, it is enough if we show the above set is invariant under automorphisms. Fix $f \in \mathrm{Aut}(\mathcal{M})$, and let $\overline{x}_1, \dots, \overline{x}_m \in i^{-1}(V)$ and $x := j\big(i(\overline{x}_1), \dots, i(\overline{x}_m)\big)$. Then

$$\begin{aligned} j\big(i(f\overline{x}_1), \dots, i(f\overline{x}_m)\big) &= \big[j \circ \gamma(f)\big]\big(i(\overline{x}_1), \dots, i(\overline{x}_m)\big) \\ &= \big[\delta\gamma(f) \circ j\big]\big(i(\overline{x}_1), \dots, i(\overline{x}_m)\big) = f(x). \end{aligned}$$

where the first equality holds by (5.1), since $\gamma(f) = I(f)$, and the second equality holds by the analogous diagram, since $\delta\big(\gamma(f)\big) = J\big(\gamma(f)\big)$. We have thus shown that the set above is invariant under an arbitrary automorphism f, concluding the proof of the claim.

Analogously, we can show that $I \circ J \sim \mathrm{id}_{\mathcal{M}'}$, so we conclude that $I : \mathcal{M} \leftrightsquigarrow \mathcal{M}' : J$ is a bi-interpretation. $\square$

Chapter 6

The Lascar Group

In this chapter we introduce the Lascar group of a complete theory T, denoted by $\mathrm{Gal_L}(T)$. The notation reflects the strong connection to (model-theoretic) Galois theory, which is discussed briefly in Subsection 6.2.1. The group was first introduced by Daniel Lascar [16] in 1982, where it is also proved that for the case of a so-called *G-compact* theory T, $\mathrm{Gal_L}(T)$ may be turned into a compact topological group. The topology was originally defined using ultraproducts. Ziegler in [27] gives a construction of the same topology without ultraproducts, and for the general case where T is not necessarily G-compact. We will use this definition of the topology.

We will assume that T admits quantifier elimination. Any complete theory T has a bi-interpretable theory that admits QE (see Section 2.4). Since we are working toward a proof of reconstructibility of T up to bi-interpretability, the assumption of QE is not a restriction.
Moreover, we assume that T has saturated models of arbitrarily large, regular cardinalities. We fix a large (i.e. $> 2^{|T|}$) saturated model $\mathbb{U} \models T$ of regular cardinality, and we define the Lascar group of T in terms of $\mathbb{U}$. It is provable under the Generalized Continuum Hypothesis that any first-order theory has arbitrarily large saturated models. In fact, by Shelah [26, Chapter VIII] it is enough to assume existence of unboundedly many cardinals κ with $\kappa^{<\kappa} = \kappa$. We choose to make this assumption, so as to avoid additional formalities and place focus on the central ideas of the Reconstruction Theorem. There are several ways to get around this assumption. For example, one can work - rather than in a fully saturated large model $\mathbb{U}$ - in a so-called $|T|^+$-*resplendent* model, which always exists, see Poizat [22, Theorem 9.15]. Casanovas and Peláez [4] shows in that the construction of the Lascar group still works in a $|T|^+$-*resplendent* model, and the results of Sections 6.1 and 6.2 still hold. Moreover, the definition of the topology, according to Ziegler [27], which we present in

Section 6.3, remains the same, since $|T|^+$-resplendent models are $|T|^+$-saturated and -homogeneous, see [4, Proposition 2.2.]. Moreover, several authors mention the use of so-called *special models*, as an alternative to saturated models, in the construction of the Lascar-group ([16], [27]).

We define in Section 6.1 the group of *Lascar-strong* automorphisms $\operatorname{Autf}_{\mathrm{L}}(\mathbb{U})$, as in [27]. We show that $\operatorname{Autf}_{\mathrm{L}}(\mathbb{U})$ is independent of the choice of $\mathbb{U}$, and can equivalently be defined as the smallest *hereditarily normal* ('very normal' in [16]) subgroup of $\operatorname{Aut}(\mathbb{U})$. These results are essentially shown in [16, Chapter 3]. We expand on these proofs, and adapt them so as to avoid using ultraproducts. In Section 6.2, we define $\operatorname{Gal}_{\mathrm{L}}(T)$ as the quotient $\operatorname{Aut}(\mathbb{U})/\operatorname{Autf}_{\mathrm{L}}(\mathbb{U})$, and show in Theorem 6.10 that this quotient is independent of the choice of $\mathbb{U}$. The result, and the idea for the proof, are again based on [16]. Finally, we define in Section 6.3 the topology on $\operatorname{Gal}_{\mathrm{L}}(T)$, according to [27], and show that $\operatorname{Gal}_{\mathrm{L}}(T)$ is a compact topological group, when it is Hausdorff.

Recall that any small (i.e. smaller than $|\mathbb{U}|$) model embeds elementarily into $\mathbb{U}$, by Theorem 2.11. Thus, we will assume that all small submodels are elementary submodels of $\mathbb{U}$. Moreover, by homogeneity of $\mathbb{U}$, any isomorphism between small submodels extends to an isomorphism of $\mathbb{U}$.

6.1 Lascar-strong Automorphisms and hereditarily normal subgroups

Definition 6.1. The group of *Lascar-strong* automorphisms is the subgroup of $\operatorname{Aut}(\mathbb{U})$ generated by all pointwise stabilizers of small submodels of $\mathbb{U}$, i.e.

$$\operatorname{Autf}_{\mathrm{L}}(\mathbb{U}) := \left\langle \bigcup \{\operatorname{Aut}_{\mathcal{M}}(\mathbb{U}) \mid \mathcal{M} \preccurlyeq \mathbb{U}\} \right\rangle .$$

For tuples $\overline{a}, \overline{b} \subset \mathbb{U}$, we say, following Kim-Pillay [14], that $\overline{a}$ and $\overline{b}$ have the same *Lascar-strong type* (write $\overline{a} \overset{\mathrm{L}}{\equiv} \overline{b}$) if there exists $f \in \operatorname{Autf}_{\mathrm{L}}(\mathbb{U})$ with $f(\overline{a}) = \overline{b}$.

For a parameterset $X \subseteq \mathbb{U}$, let $\mathbb{U}_X$ denote the $\mathcal{L}_X$ structure on $\mathbb{U}$. Then we denote the group of Lascar-strong automorphisms of $\mathbb{U}_X$ by $\operatorname{Autf}_{\mathrm{L}}(\mathbb{U}/X)$, i.e.

$$\operatorname{Autf}_{\mathrm{L}}(\mathbb{U}/X) = \left\langle \bigcup \{\operatorname{Aut}_{\mathcal{M}}(\mathbb{U}) \mid X \subseteq \mathcal{M} \preccurlyeq \mathbb{U}\} \right\rangle ,$$

and we write $\overline{a} \overset{\mathrm{L}}{\equiv}_X \overline{b}$ if $\overline{a}$ and $\overline{b}$ have the same Lascar-strong type over X.

Note that all stabilizers $\operatorname{Aut}_{\mathcal{M}}(\mathbb{U})$ are normal subgroups of $\operatorname{Aut}(\mathbb{U})$, and therefore the $\operatorname{Autf}_{\mathrm{L}}(\mathbb{U})$ forms a normal subgroup of $\operatorname{Aut}(\mathbb{U})$.

Example 6.2. If T admits Skolem functions, then all automorphisms of $\mathbb{U}$ are Lascar-strong, since the smallest substructure of $\mathbb{U}$ (that is, the substructure made up of the interpretations of all terms) is itself a model of T.

Definition 6.3. For automorphisms $f, g \in \mathrm{Aut}(\mathbb{U})$, we say that f and g are *heredetarily conjugate* or *h-conjugate*, and write $f \overset{\mathrm{h}}{\sim} g$, if there exists a model $\mathbb{U}' \succcurlyeq \mathbb{U}$ such that f, g can be extended to $f', g' \in \mathrm{Aut}(\mathbb{U}')$ with $f' = h'g'h'^{-1}$ for some $h' \in \mathrm{Aut}(\mathbb{U}')$. We say that a subgroup $G \leq \mathrm{Aut}(\mathbb{U})$ is *hereditarily normal* or *h-normal* if it is closed under h-conjugation, i.e. for any $f \in G$, and $g \in \mathrm{Aut}(\mathbb{U})$ with $g \overset{\mathrm{h}}{\sim} f$, we have $g \in G$.

Clearly h-conjugacy is an equivalence relation. Morover, it is easy to see that intersections of h-normal subgroups are h-normal. Thus, there exists a smallest h-normal subgroup of $\mathrm{Aut}(\mathbb{U})$, namely, the intersection of all h-normal subgroups. We will denote it by $H(\mathbb{U})$ (see '$\Gamma(\mathbb{U})$' in [16]). Denote the h-conjugacy class of the identity by $I_h(\mathbb{U})$ (see '$A(\mathbb{U})$' in [16]. We aim to prove that $H(\mathbb{U}) = \mathrm{Autf_L}(\mathbb{U}) = \langle I_h(\mathbb{U}) \rangle$.
For a subset $X \subseteq \mathbb{U}$, we can relativize $H(\mathbb{U})$ and $I_h(\mathbb{U})$ to X: We denote the smallest h-normal subgroup of $\mathrm{Aut}(\mathbb{U}_X)$ by $H(\mathbb{U}/X)$. Similarly, we write $I_h(\mathbb{U}/X)$ for the h-conjugacy class of the identity in $\mathrm{Aut}(\mathbb{U}_X)$.

We will generally formulate results for the general case where a finite parameterset X is present, but assume in proofs that $X = \emptyset$. This does not reduce generality, since we can simply add X to our language in the form of constants. Moreover, in instances where we assume $\aleph_0$-categoricity, we will only consider finite X, so that T_X is still $\aleph_0$-categorical. Note however, that unless $X \subseteq dcl^{eq}(\emptyset)$, the theory T_X will not usually be bi-interpretable with T. The role of X will become relevant in the proof of the reconstruction theorem.

The following proof expands upon the proof of Proposition 32 in Lascar's original paper [16]. In his proof, Lascar uses ultrafilters, which we avoid here, by a compactness argument.

Proposition 6.4. *For any model $X \subseteq \mathcal{M} \preccurlyeq \mathbb{U}$,*

$$\mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \subseteq I_h(\mathbb{U}/X).$$

Proof. Assume w.l.o.g. that $X = \emptyset$.
Let $\mathcal{M} \preccurlyeq \mathbb{U}$ and $f \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$.
Extend $\mathcal{L}$ by two unary function symbols, F and σ, and one unary relation symbol P, and denote the expanded language by $\mathcal{L}'$. Then consider the set Σ of $\mathcal{L}'_{\mathbb{U}}$-sentences stating that F is an $\mathcal{L}$-automorphism that fixes P pointwise, P defines an $\mathcal{L}$-submodel,

$\mathbb{U}$ is a submodel, and σ is an elementary map from $\mathbb{U}$ to P, i.e. $\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$ where

$$\begin{aligned}
\Sigma_1 :=& \big\{ \forall \overline{v} : \ \varphi(\overline{v}) \leftrightarrow \varphi(F\overline{v}) \ \big| \ \varphi \in \mathcal{L} \big\} \ \cup \ \big\{ \forall v \exists w : \ F(w) = v \big\} \ \cup \ \big\{ \forall v : \ P(v) \to F(v) = v \big\} \\
\Sigma_2 :=& \big\{ \forall \overline{v} \exists \overline{w} : \ P(\overline{v}, \overline{w}) \wedge \varphi(\overline{v}, \overline{w}) \ \big| \ \varphi \in \mathcal{L} \text{ and } T \vdash \forall \overline{v} \exists \overline{w} \ \varphi(\overline{v}, \overline{w}) \big\} \\
\Sigma_3 :=& \big\{ \alpha(c_{\overline{x}}) = c_{\alpha^{\mathbb{U}}(\overline{x})} \ \big| \ \alpha \text{ function symbol in } \mathcal{L} \text{ and } \overline{x} \subset \mathbb{U} \big\} \\
& \cup \big\{ R(c_{\overline{x}}) \ \big| \ R \text{ relation symbol in } \mathcal{L} \text{ and } \overline{x} \in R(\mathbb{U}) \big\} \\
& \cup \big\{ d = c_{d^{\mathbb{U}}} \ \big| \ d \text{ constant symbol in } \mathcal{L} \big\} \\
\Sigma_4 :=& \big\{ P\big(\sigma(c_x)\big) \ \big| \ x \in \mathbb{U} \big\} \cup \big\{ \varphi\big(\sigma(c_{\overline{x}})\big) \ \big| \ \overline{x} \subset \mathbb{U} \text{ and } \mathbb{U} \models \varphi(\overline{x}) \big\}.
\end{aligned}$$

We will show $\Sigma \cup T$ is finitely satisfiable. Notice firstly that since f fixes $\mathcal{M} \preccurlyeq \mathbb{U}$ pointwise by assumption,

$$\big\langle \mathbb{U}, f, \mathcal{M} \big\rangle \models \Sigma_1 \cup \Sigma_2 \cup \Sigma_3.$$

Now, let $\overline{x} \subset \mathbb{U}$ be a finite tuple in $\mathbb{U}$, and let $\varphi(\overline{v}) \in \mathrm{tp}(\overline{x})$. Then

$$\mathbb{U} \models \exists \overline{v} \ \varphi(\overline{v})$$

and since $\mathcal{M} \preccurlyeq \mathbb{U}$,

$$\mathcal{M} \models \exists \overline{v} \ \varphi(\overline{v}).$$

Thus, we can find a tuple $\overline{y} \subset \mathcal{M}$ of the same length with $\mathbb{U} \models \varphi(\overline{y})$, and define a function $\sigma : \mathbb{U} \to \mathbb{U}$ with $\sigma(\overline{x}) = \overline{x}$ pointwise. Then

$$\big\langle \mathbb{U}, f, \sigma, \mathcal{M} \big\rangle \models \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \big\{ P\big(\sigma(c_{\overline{x}})\big), \ \varphi\big(\sigma(c_{\overline{x}})\big\}.$$

We conclude that $\Sigma \cup T$ is finitely satisfiable, so by Compactness, we can find $\mathbb{U}' \succcurlyeq \mathbb{U}$ with $\mathcal{M}' \preccurlyeq \mathbb{U}'$, $f' \in \mathrm{Aut}_{\mathcal{M}'}(\mathbb{U}')$ and $\sigma : \mathbb{U}' \to \mathbb{U}'$ s.t.

$$\big\langle \mathbb{U}', f', \sigma, \mathcal{M}' \big\rangle \models \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4.$$

Moreover, since arbitrarily large saturated models exist, we may assume that $\mathbb{U}'$ is saturated (simply find a large saturated model $\mathbb{U}^+ \succcurlyeq \mathbb{U}'$, and extend f' by homogeneity). Now, let $\mathbb{U}_1 := \sigma(\mathbb{U}) \preccurlyeq \mathcal{M}'$. By construction of Σ_4, $\sigma : \mathbb{U} \to \mathbb{U}_1$ is an $\mathcal{L}$-isomorphism. Moreover, by Σ_1, we get $f'|_{\mathbb{U}} = f$, and $f'|_{\mathcal{M}'} = \mathrm{id}_{\mathcal{M}'}$, so in particular, $f'|_{\mathbb{U}_1} = \mathrm{id}_{\mathbb{U}_1}$. We conclude that

$$f \subseteq f' = \sigma \circ (\sigma^{-1} f' \sigma) \circ \sigma^{-1}$$

and $(\sigma^{-1} f' \sigma)|_{\mathbb{U}} = \mathrm{id}_{\mathbb{U}}$, so $f \overset{\mathrm{h}}{\sim} \mathrm{id}_{\mathbb{U}}$, i.e. by definition $f \in I_h(\mathbb{U})$. $\square$

Since $\mathrm{Autf_L}(\mathbb{U}/X)$ is generated by the union of all stabilizers $\mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$, we immediately get the following Corollary.

Corollary 6.5. $\mathrm{Autf_L}(\mathbb{U}/X) \subseteq \big\langle I_h(\mathbb{U}/X)\big\rangle$.

Recall that one aim of this section is to prove that $\mathrm{Autf_L}(\mathbb{U}/X) = H(\mathbb{U}/X)$.
It follows immediately by definition that $\langle I_h(\mathbb{U}/X)\rangle \subseteq H(\mathbb{U}/X)$. Together with the above corollary, we get

$$\mathrm{Autf_L}(\mathbb{U}/X) \subseteq \big\langle I_h(\mathbb{U}/X)\big\rangle \subseteq H(\mathbb{U}/X). \tag{6.1}$$

Thus, we only need to show that $H(\mathbb{U}/X) \subseteq \mathrm{Autf_L}(\mathbb{U}/X)$. We will do so by proving that $\mathrm{Autf_L}(\mathbb{U}/X)$ is h-normal.

The following proposition asserts that $\mathrm{Autf_L}(\mathbb{U})$ is - up to extending and restricting automorphisms - independent of the choice of large saturated model $\mathbb{U}$.

Proposition 6.6. *Suppose* $\mathbb{U}' \succcurlyeq \mathbb{U}$ *is saturated, and* $f \in \mathrm{Aut}(\mathbb{U})$, $f' \in \mathrm{Aut}_X(\mathbb{U}')$ *with* $f \subseteq f'$. *Then*

$$f \in \mathrm{Autf_L}(\mathbb{U}/X) \iff f' \in \mathrm{Autf_L}(\mathbb{U}'/X).$$

Proof. We may assume w.l.o.g. that $X = \emptyset$. Moreover, we assume that $\mathbb{U}'$ has higher cardinality than $\mathbb{U}$. If not, we can further extend $\mathbb{U}'$ to a saturated model $\mathbb{U}^+ \succcurlyeq \mathbb{U}'$ of higher cardinality, and extend f' to $f^+ \in \mathrm{Aut}(\mathbb{U}^+)$, and apply the proposition first to get $f \in \mathrm{Autf_L}(\mathbb{U}) \iff f^+ \in \mathrm{Autf_L}(\mathbb{U}^+)$, and again to get $f' \in \mathrm{Autf_L}(\mathbb{U}') \iff f^+ \in \mathrm{Autf_L}(\mathbb{U}^+)$.

Assume first that $f \in \mathrm{Autf_L}(\mathbb{U})$. That is, there exist $\mathcal{M}_i \preccurlyeq \mathbb{U}$ and $f_i \in \mathrm{Aut}_{\mathcal{M}_i}(\mathbb{U})$ for $i = 1, \dots, n$, such that $f = f_1 \circ \dots \circ f_n$. By homogeneity, we may extend each f_i to some $f_i' \in \mathrm{Aut}_{\mathcal{M}_i}(\mathbb{U}')$, and get $f'|_{\mathbb{U}} = (f_1' \circ \dots \circ f_n')|_{\mathbb{U}}$. Then clearly $f_1' \circ \dots \circ f_n' \in \mathrm{Autf_L}(\mathbb{U}')$, so we conclude

$$f' = \underbrace{(f' \circ (f_1' \circ \cdots \circ f_n')^{-1})}_{\in \mathrm{Aut}_{\mathbb{U}}(\mathbb{U}')} \circ (f_1' \circ \cdots \circ f_n') \in \mathrm{Autf_L}(\mathbb{U}').$$

Assume now that $f' \in \mathrm{Autf_L}(\mathbb{U}')$. Find $n \in \omega$, and for each $i = 1, \dots, n$ a model $\mathcal{M}_i' \preccurlyeq \mathbb{U}'$ and an automorphism $f_i' \in \mathrm{Aut}_{\mathcal{M}_i'}(\mathbb{U}')$ with $f' = f_1' \circ \dots \circ f_n'$. We may assume, in light of the downward Löwenheim-Skolem Theorem, that $|\mathcal{M}_i'| < |\mathbb{U}|$ for all $i = 1, \dots, n$. Now, fix a small model $\mathcal{M} \preccurlyeq \mathbb{U}$. Since

$$\big|\mathcal{M} \cup f(\mathcal{M}) \cup \mathcal{M}_1' \cup \cdots \cup \mathcal{M}_n'\big| < \big|\mathbb{U}\big|,$$

we can find by Lemma 2.14 a saturated model $\tilde{\mathbb{U}} \preccurlyeq \mathbb{U}'$ with $|\tilde{\mathbb{U}}| = |\mathbb{U}|$, and

$$\mathcal{M} \cup f(\mathcal{M}) \cup \mathcal{M}'_1 \cup \cdots \cup \mathcal{M}'_n \subseteq \tilde{\mathbb{U}},$$

such that $\tilde{f}_i := (f'_i)|_{\tilde{\mathbb{U}}} \in \mathrm{Aut}(\tilde{\mathbb{U}})$ for all $i = 1, \ldots, n$. Since $\tilde{\mathbb{U}}$ and $\mathbb{U}$ are both saturated models, containing the small models $\mathcal{M}$ and $f(\mathcal{M})$, we can find an isomorphism $\gamma : \tilde{\mathbb{U}} \to \mathbb{U}$ which fixes $\mathcal{M}$ and $f(\mathcal{M})$ pointwise. Now, setting $\mathcal{M}_i := \gamma(\mathcal{M}'_i)$, we get

$$\gamma \circ (f'_i)|_{\tilde{\mathbb{U}}} \circ \gamma^{-1} \in \mathrm{Aut}_{\gamma(\mathcal{M}'_i)}(\mathbb{U})$$

for each $i = 1, \ldots, n$. Thus, we see that

$$\gamma \circ \tilde{f} \circ \gamma^{-1} = \left(\gamma \circ (f'_1)|_{\tilde{\mathbb{U}}} \circ \gamma^{-1}\right) \circ \cdots \circ \left(\gamma \circ (f'_n)|_{\tilde{\mathbb{U}}} \circ \gamma^{-1}\right) \in \mathrm{Autf_L}(\mathbb{U}).$$

Finally, note that

$$\left(\gamma \circ \tilde{f} \circ \gamma^{-1}\right)|_{\mathcal{M}} = f|_{\mathcal{M}}$$

since γ fixes $\mathcal{M}$ and $f(\mathcal{M})$ pointwise, and $\tilde{f}|_{\mathcal{M}} = (f')|_{\mathcal{M}} = f|_{\mathcal{M}}$. We conclude that $f \in \mathrm{Autf_L}(\mathbb{U})$. □

It is now not difficult to show that $\mathrm{Autf_L}(\mathbb{U}/X)$ is indeed h-normal.

Proposition 6.7. $\mathrm{Autf_L}(\mathbb{U}/X)$ *is an h-normal subgroup of* $\mathrm{Aut}(\mathbb{U})$.

Proof. Assume again that $X = \emptyset$.
Firstly, it is easy to see that $\mathrm{Autf_L}(\mathbb{U})$ is normal: Let $f_1, \ldots, f_n \in \mathrm{Aut}(\mathbb{U})$ be automorphisms that each fix some submodel of $\mathbb{U}$ pointwise. Then for any $h \in \mathrm{Aut}(\mathbb{U})$,

$$h^{-1} \circ (f_1 \circ \ldots \circ f_n) \circ h = (h^{-1} f_1 h) \circ \ldots \circ (h^{-1} f_n h)$$

and clearly, if f_i fixes $\mathcal{M}_i \preccurlyeq \mathbb{U}$ pointwise, then $h^{-1} f_i h$ fixes $h^{-1}(\mathcal{M}_i) \preccurlyeq \mathbb{U}$ pointwise.

Suppose now that $f \in \mathrm{Autf_L}(\mathbb{U})$, and $g \in \mathrm{Aut}(\mathbb{U})$ with $f \overset{\mathrm{h}}{\sim} g$. Then by definition, there exists a model $\mathbb{U}' \succcurlyeq \mathbb{U}$ and extensions $f', g' \in \mathrm{Aut}(\mathbb{U}')$ of f, respectively g, such that with $g' = \gamma^{-1} \circ f' \circ \gamma$ for some $\gamma \in \mathrm{Aut}(\mathbb{U}')$. By Proposition 6.6, $f' \in \mathrm{Autf_L}(\mathbb{U}')$, so by normality of $\mathrm{Autf_L}(\mathbb{U}')$, we get $g' \in \mathrm{Autf_L}(\mathbb{U}')$. Again by Proposition 6.6, we conclude that$g \in \mathrm{Autf_L}(\mathbb{U})$. □

Since $H(\mathbb{U}/X)$ is the intersection of all h-normal subgroups of $\mathrm{Aut}_X(\mathbb{U})$, the proposition we have just proved yields

$$H(\mathbb{U}/X) \subseteq \mathrm{Autf_L}(\mathbb{U}/X),$$

and by (6.1), we get the following corollary.

Corollary 6.8. $\mathrm{Autf_L}(\mathbb{U}/X) = \langle I_h(\mathbb{U}/X) \rangle = H(\mathbb{U}/)$.

6.2 The Group

Definition 6.9. The *Lascar-group* of the model $\mathbb{U}$ is

$$\mathrm{Gal_L}(\mathbb{U}) := \mathrm{Aut}(\mathbb{U})/\,\mathrm{Autf_L}(\mathbb{U}).$$

For $X \subset \mathbb{U}$, we define the relativized Lascar-group of $\mathbb{U}$ as

$$\mathrm{Gal_L}(\mathbb{U}/X) := \mathrm{Aut}_X(\mathbb{U})/\,\mathrm{Autf_L}(\mathbb{U}/X).$$

We want to define the Lascar group as an object associated with our theory T, rather than the model $\mathbb{U}$. To do so, we prove that $\mathrm{Gal_L}(\mathbb{U}/X)$ is independent of the choice of the large saturated model $\mathbb{U}$. The proof we give expands upon the Lascar [16, Corollary 42].

Theorem 6.10. *Let $\mathbb{U}$ and $\mathbb{U}'$ both be large saturated models of T, containing X. Then $\mathrm{Gal_L}(\mathbb{U}/X)$ and $\mathrm{Gal_L}(\mathbb{U}'/X)$ are isomorphic as groups.*

Proof. Assume that $X = \emptyset$.
We may assme w.l.o.g. that $\mathbb{U}'$ has larger cardinality than $\mathbb{U}$, and $\mathbb{U} \preccurlyeq \mathbb{U}'$. Otherwise, we can find a model $\mathbb{U}^+$ of larger cardinality than both $\mathbb{U}$ and $\mathbb{U}'$, which is an elementary extension of both, and apply the theorem twice.

Since we assumed that $\mathbb{U}'$ is saturated and larger than $\mathbb{U}$, we can find for any automorphism of $\mathbb{U}$, an extension in $\mathrm{Aut}(\mathbb{U}')$. Using the axiom of choice, we fix for each $f \in \mathrm{Aut}(\mathbb{U})$ such an extension $f' \in \mathrm{Aut}(\mathbb{U}')$. Then, we define

$$\rho : \begin{cases} \mathrm{Aut}(\mathbb{U}) \to \mathrm{Gal_L}(\mathbb{U}') \\ f \mapsto f' \cdot \mathrm{Autf_L}(\mathbb{U}'). \end{cases}$$

<u>Claim:</u> ρ is a group homomorphism.
Let $f, g \in \mathrm{Aut}(\mathbb{U})$. We want to show that $(fg)' \equiv f'g' \mod \mathrm{Autf_L}(\mathbb{U}')$. This follows immediately from $\big((fg)' \circ (f'g')^{-1}\big)|_{\mathbb{U}} = \mathrm{id}_{\mathbb{U}} \in \mathrm{Autf_L}(\mathbb{U})$, together with Proposition6.6.
<u>Claim:</u> $\rho^{-1}(\{\mathrm{Autf_L}(\mathbb{U}')\}) = \mathrm{Autf_L}(\mathbb{U})$.
This is precisely the statement of Proposition 6.6.
<u>Claim:</u> ρ is surjective.
Fix an arbitrary automorphism $g \in \mathrm{Aut}(\mathbb{U}')$. We will show that $g \in f' \cdot \mathrm{Autf_L}(\mathbb{U}')$ for some $f \in \mathrm{Aut}(\mathbb{U})$.
By downward Löwenheim-Skolem, find any small model $\mathcal{M} \preccurlyeq \mathbb{U}$. Then by saturation of $\mathbb{U}$, find $\mathcal{N} \preccurlyeq \mathbb{U}$ with

$$\mathrm{tp}^{\mathbb{U}}(\overline{\mathcal{N}}/\mathcal{M}) = \mathrm{tp}^{\mathbb{U}'}(g(\overline{\mathcal{M}})/\mathcal{M}).$$

Since $\mathbb{U} \preccurlyeq \mathbb{U}'$, we have $\mathrm{tp}^{\mathbb{U}'}(\overline{\mathcal{N}}/\mathcal{M}) = \mathrm{tp}^{\mathbb{U}}(\overline{\mathcal{N}}/\mathcal{M})$, so there exists by homogeneity of $\mathbb{U}'$, an automorphism $h \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}')$ with $h(g(\overline{\mathcal{M}})) = \overline{\mathcal{N}}$. Moreover, since

$$\mathrm{tp}^{\mathbb{U}}(\overline{\mathcal{N}}) = \mathrm{tp}^{\mathbb{U}'}(\overline{\mathcal{N}}) = \mathrm{tp}^{\mathbb{U}'}(g(\overline{\mathcal{M}})) = \mathrm{tp}^{\mathbb{U}'}(\overline{\mathcal{M}}) = \mathrm{tp}^{\mathbb{U}}(\overline{\mathcal{M}}),$$

we can find $f \in \mathrm{Aut}(\mathbb{U})$ with $f(\overline{\mathcal{M}}) = \overline{\mathcal{N}}$. Then f' agrees with $h \circ g$ on $\mathcal{M}$, which agrees with g on $\mathcal{M}$, since $h|_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$. Hence, we get

$$g \in f' \cdot \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}') = \rho(f).$$

We have shown that ρ is a surjective group homomorphism with $\rho^{-1}(\{\mathrm{Autf}_{\mathrm{L}}(\mathbb{U}')\}) = \mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$. It follows that

$$\tilde{\rho} : \begin{cases} \mathrm{Gal}_{\mathrm{L}}(\mathbb{U}) \to \mathrm{Gal}_{\mathrm{L}}(\mathbb{U}') \\ f \cdot \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}) \mapsto f' \cdot \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}') \end{cases}$$

is a well-defined, bijective group homomorphism. □

In light of the above theorem, we can now sensibly define the Lascar-group of the theory T, denoted by $\mathrm{Gal}_{\mathrm{L}}(T)$, as

$$\mathrm{Gal}_{\mathrm{L}}(T) := \mathrm{Gal}_{\mathrm{L}}(\mathbb{U}).$$

Naturally, for a parameterset $X \subset \mathbb{U}$, we define the relativized Lascar-group $\mathrm{Gal}_{\mathrm{L}}(T/X)$ to be $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U}/X)$.

Example 6.11. Recall Example 6.2 in Section 6.1: If T admits Skolem functions, then all automorphisms of $\mathbb{U}$ are Lascar-strong, and therefore, $\mathrm{Gal}_{\mathrm{L}}(T)$ is trivial.

Example 6.12 ([16])**.** Similarly, if T is the theory of infinite sets in the language of equality, then $\mathrm{Gal}_{\mathrm{L}}(T)$ is trivial: Let M be an uncountable set and $f : M \to M$ a bijection. Let $N \supseteq M$ be any set such that $|N \setminus M|$ is infinite. We can extend f to a bijection $f' : N \to N$ by simply leaving all elements of $N \setminus M$ fixed. Then f' is Lascar-strong, so f is as well by Proposition 6.6.

6.2.1 Connection to Model-Theoretic Galois Theory

This section is intended purely as an aside, to give motivation for the notation $\mathrm{Gal}_{\mathrm{L}}(T)$, which hints at the connection of the Lascar-group to (model-theoretic) Galois theory.

Let $\mathcal{L}$ be a language, and $\mathbb{U}$ a saturated $\mathcal{L}$-structure. Given a set $X \subseteq \mathbb{U}$ and an element $a \in \mathbb{U}$, recall that we say that a is *definable* over X if there is an $\mathcal{L}_X$-formula φ such that a is the unique element in $\varphi(\mathbb{U}_X)$. What is more, following Shelah [26, Section III.6], we say that a is *algebraic* over X if there exists an $\mathcal{L}_X$-formula φ such that a is one of finitely many elements in $\varphi(\mathbb{U}_X)$. We denote by $\mathrm{dcl}(X)$ the set of all elements of $\mathbb{U}$ that are definable over X, and call this the *definitional closure* of X. Similarly, we define the *algebraic closure* $\mathrm{acl}(X)$ as the set of all elements of $\mathbb{U}$ that are algebraic over X.

It is not difficult to see that since $\mathbb{U}$ is saturated, $\mathrm{dcl}(X)$ is precisely the set of elements of $\mathbb{U}$ that are fixed by $\mathrm{Aut}_X(\mathbb{U})$, while $\mathrm{acl}(X)$ is the set of elements which have finite orbit under $\mathrm{Aut}_X(\mathbb{U})$. These orbits are the sets $\varphi(\mathbb{U})$, where φ is an $\mathcal{L}_X$-formula with finitely many solutions.
We consider now the subgroup of the symmetric group $\mathrm{Sym}(\mathrm{acl}(X))$ obtained by restricting automorphisms $f \in \mathrm{Aut}_X(\mathbb{U})$, i.e.

$$G(\mathrm{acl}(X)/X) := \{f|_{\mathrm{acl}(X)} \mid f \in \mathrm{Aut}_X(\mathbb{U})\} \simeq \mathrm{Aut}(\mathbb{U}/X)/\,\mathrm{Aut}_{\mathrm{acl}(X)}(\mathbb{U}).$$

We call this the *absolute Galois group* of X ([22]).

The following example serves as the bridge between 'model-theoretic' Galois theory and classical Galois theory.

Example 6.13. Consider the theory ACF_0 of algebraically closed fields of characteristic zero, with the complex numbers $\mathbb{C}$ as a saturated model, and let $F \leq \mathbb{C}$ be any subfield. Then the model-theoretic algebraic closure of F in $\mathbb{C}$, as defined above, is precisely the algebraic closure in the sense of algebra, and the model-theoretic absolute Galois group of F, as defined above, is precisely the absolute Galois group of F in the sense of algebra.
Moreover, the Lascar-strong automorphisms of $\mathbb{C}$ over F are precisely the field isomorphisms of $\mathbb{C}$ which fix $\mathrm{acl}(F)$, i.e.

$$\mathrm{Autf}_{\mathrm{L}}(\mathbb{C}/F) = \mathrm{Aut}_{\mathrm{acl}(F)}(\mathbb{C}).$$

The inclusion from right to left is clear, since $\mathrm{acl}(F)$ is a model of ACF, containing F. The other inclusion is similarly trivial, since any model of ACF containing F contains $\mathrm{acl}(F)$.
We conclude therefore that the absolute Galois group of F, in this case, is precisely the Lascar group $\mathrm{Gal}_{\mathrm{L}}(\mathbb{C}/F)$.

6.3 The Topological Group

Recall that the autormorphism group $\mathrm{Aut}(\mathbb{U})$ of $\mathbb{U}$ becomes a topological group when equipped with the *topology of pointwise-convergence*. We will now work toward a definition of a topology on $\mathrm{Gal}_{\mathrm{L}}(T)$, following [27, Chapter 4].

Fix two small submodels $\mathcal{M}, \mathcal{N} \preccurlyeq \mathbb{U}$. Then for any $f \in \mathrm{Aut}(\mathbb{U})$, the class of f in $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ depends only on the type of $f(\mathcal{M})$ over $\mathcal{N}$.

Lemma 6.14 ([27, Lemma 1])**.** *For any two automorphisms $f, g \in \mathrm{Aut}(\mathbb{U})$ with $\mathrm{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big) = \mathrm{tp}\big(g(\overline{\mathcal{M}})/\mathcal{N}\big)$, we get*

$$f \equiv g \mod \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}).$$

Proof. Assume that $\mathrm{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big) = \mathrm{tp}\big(g(\overline{\mathcal{M}})/\mathcal{N}\big)$. Then by homogeneity of $\mathbb{U}$, we can find $h \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ with $h\big(f(\overline{\mathcal{M}})\big) = g(\overline{\mathcal{M}})$. Thus, we get

$$h \circ f \equiv g \mod \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}),$$

and since $h \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}) \subseteq \mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$, we conclude that $f \equiv g \mod \mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$. $\square$

Remark 6.15. *By the Lemma we have just shown, if we choose $\mathcal{M}, \mathcal{N}$ to have cardinality $|\mathcal{L}|$ (by downward Löwenhem-Skolem), then it follows that*

$$|\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})| \leq |S_{\mathcal{M}}(\mathcal{N})| \leq 2^{|T| \cdot |\mathcal{M}| \cdot |\mathcal{N}|} = 2^{|T|}.$$

We denote by $S_{\mathcal{M}}(\mathcal{N})$ the set of types $\mathrm{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big)$, where $f \in \mathrm{Aut}(\mathbb{U})$, and by $\sigma : \mathrm{Aut}(\mathbb{U}) \to S_{\mathcal{M}}(\mathcal{N})$ the map $\sigma(f) := \mathrm{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big)$.

Consider the quotient map $\pi : \mathrm{Aut}(\mathbb{U}) \to \mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$. From Lemma 6.14, it follows that there exists a map $\tau : S_{\mathcal{M}}(\mathcal{N}) \to \mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ such that π factors through $S_{\mathcal{M}}(\mathcal{N})$ via τ, as in the following diagram.

$$\begin{tikzcd} \mathrm{Aut}(\mathbb{U}) \arrow[r, "\pi"] \arrow[d, "\sigma"'] & \mathrm{Gal}_{\mathrm{L}}(\mathbb{U}) \\ S_{\mathcal{M}}(\mathcal{N}) \arrow[ur, dashed, "\tau"'] & \end{tikzcd} \tag{6.2}$$

Now, notice that $S_{\mathcal{M}}(\mathcal{N})$ carries the subspace-topology of the topology on $S_{|\mathcal{M}|}(\mathcal{N})$, i.e. the topology generated by the clopen sets

$$\Big\{p \in S_{|\mathcal{M}|}(\mathcal{N}) \;\Big|\; \varphi \in p\Big\}, \text{ where } \varphi \in \mathcal{L}_{\mathcal{N}}.$$

Thus, we can equip $\mathrm{Gal_L}(\mathbb{U})$ with the quotient topology of τ. That is, the topology whose open sets are precisely those $A \subseteq \mathrm{Gal_L}(\mathbb{U})$ whose preimage $\tau^{-1}(A)$ is open in $S_{\mathcal{M}}(\mathcal{N})$.

Remark 6.16. *Note that the map* $\sigma : \mathrm{Aut}(\mathbb{U}) \to S_{\mathcal{M}}(\mathcal{N})$ *is continuous: If* $\varphi \in \mathcal{L}_{\mathcal{N}}(\overline{v})$ *is a formula with parameters* $\overline{y} \in \mathcal{N}^{<\omega}$*, then for any* $f, g \in \mathrm{Aut}(\mathbb{U})$ *with* $f(\overline{y}) = g(\overline{y})$*, clearly* $\mathrm{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big)$ *and* $\mathrm{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big)$ *agree on* φ*. Therefore, the projection* $\pi : \mathrm{Aut}(\mathbb{U}) \to \mathrm{Gal_L}(\mathbb{U})$ *is continuous under the topology on* $\mathrm{Gal_L}(\mathbb{U})$ *defined above.*

We prove now that this topology on $\mathrm{Gal_L}(T)$ does not depend on the choice of small submodels $\mathcal{M}$ and $\mathcal{N}$.

Lemma 6.17 ([27])**.** *Let* $\mathcal{M}', \mathcal{N}' \preccurlyeq \mathbb{U}$ *be small models,* $\sigma' : \mathrm{Aut}(\mathbb{U}) \to S_{\mathcal{M}'}(\mathcal{N}')$ *the map* $f \mapsto \mathrm{tp}\big(f(\overline{\mathcal{M}}')/\mathcal{N}'\big)$*, and* $\tau' : S_{\mathcal{M}'}(\mathcal{N}') \to \mathrm{Gal_L}(\mathbb{U})$ *such that* $\tau' \circ \sigma' = \pi$*. Then the quotient topology under* τ' *on* $\mathrm{Gal_L}(\mathbb{U})$ *is equal to the quotient topology under* τ*.*

Proof. The situation of the lemma is as in the diagram below.

$$\begin{tikzcd}
 & S_{\mathcal{M}}(\mathcal{N}) \arrow[dr, "\tau"] & \\
\mathrm{Aut}(\mathbb{U}) \arrow[ur, "\sigma"] \arrow[rr, "\pi"] \arrow[dr, "\sigma'"'] & & \mathrm{Gal_L}(\mathbb{U}). \\
 & S_{\mathcal{M}'}(\mathcal{N}') \arrow[ur, "\tau'"'] &
\end{tikzcd}$$

We can assume w.l.o.g. that $\mathcal{M} \subseteq \mathcal{M}'$ and $\mathcal{N} \subseteq \mathcal{N}'$. If not, we can find, by saturation of $\mathbb{U}$, models $\mathcal{M}_1$ and $\mathcal{N}_1$ containing $\mathcal{M} \cup \mathcal{M}'$, respectively $\mathcal{N} \cup \mathcal{N}'$, and then apply the lemma twice. Then, there exists a natural projection $\iota : S_{\mathcal{M}'}(\mathcal{N}') \to S_{\mathcal{M}}(\mathcal{N})$, mapping $\mathrm{tp}\big(f(\overline{\mathcal{M}}')/\mathcal{N}'\big)$ to $\mathrm{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big)$. Then, it is clear that the following diagram commutes.

$$\begin{tikzcd}
 & S_{\mathcal{M}}(\mathcal{N}) \arrow[dr, "\tau"] & \\
S_{\mathcal{M}'}(\mathcal{N}') \arrow[ur, "\iota"] \arrow[rr, "\tau'"'] & & \mathrm{Gal_L}(\mathbb{U})
\end{tikzcd}$$

Now, since ι is a continuous surjective map, and since $S_{\mathcal{M}'}(\mathcal{N}')$ is compact and $S_{\mathcal{M}}(\mathcal{N})$ is Hausdorff, we conclude that for any $A \subseteq S_{\mathcal{M}}(\mathcal{N})$ we have

$$A \text{ is open in } S_{\mathcal{M}}(\mathcal{N}) \iff \iota^{-1}(A) \text{ is open in } S_{\mathcal{M}'}(\mathcal{N}').$$

Therefore, the final topology on $\mathrm{Gal_L}(\mathbb{U})$ under τ is precisely the final topology under $\tau \circ \iota$, which by the diagram above, is the same as the final topology under τ'. $\square$

From here onward, we may consider $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ as a topologized group, under the above defined topology. It is easy to see that $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ is compact.

Proposition 6.18. $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ *is compact.*

Proof. This follows immediately from the fact that $S_{\mathcal{M}}(\mathcal{N})$ is compact as a closed subspace of the compact space $S_{|\mathcal{M}|}(\mathcal{N})$ and that continuous images of compact sets are compact. $\square$

Lascar [16] shows that $\mathrm{Gal}_{\mathrm{L}}(T)$ is a topological group, (i.e. multiplication and inversion are continuous), for theories where $\mathrm{Gal}_{\mathrm{L}}(T)$ is Hausdorff. This was improved by Ziegler, who shows in [27] that $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ is in fact a topological group for any first-order theory T. Since Lascar's Reconstruction Theorem is concerned only with those theories that have Hausdorff Lascar group (*G-compact* theories), we give here the proof that $\mathrm{Gal}_{\mathrm{L}}(T)$ is a topological group only for that case. We follow [27, Lemma 17].

Proposition 6.19. *If* $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ *is Hausdorff, then it is a topological group.*

Proof. Let $\mathcal{M}, \mathcal{N}$ and σ, τ be as in (6.2). We define

$\mathcal{P} := \big\{ \big(\sigma(f), \sigma(g), \sigma(fg)\big) \bigm| f, g \in \mathrm{Aut}(\mathbb{U}) \big\}$ and $\mathcal{I} := \big\{ \big(\sigma(f), \sigma(f^{-1})\big) \bigm| f \in \mathrm{Aut}(\mathbb{U}) \big\}$.

Then, the graph of multiplication on $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ is the projection

$$\tau(\mathcal{P}) = \Big\{ \big(\pi(f), \pi(g), \pi(fg)\big) \mid f, g \in \mathrm{Aut}(\mathbb{U}) \Big\}$$

and analogously for inversion.

We will show that $\mathcal{P}$ and $\mathcal{I}$ are closed in $S_{\mathcal{M}}(\mathcal{N})$. Since $S_{\mathcal{M}}(\mathcal{N})$ is compact and $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ is Hausdorff, and τ is continuous, it will follow that the graphs of multiplication and inversion on $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ are closed as well. By compactness of $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ (see Proposition 6.18), any function on $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ with a closed graph is continuous, so we will conclude that multiplication and inversion are continuous.

Claim: $\mathcal{P}$ and $\mathcal{I}$ are closed in $S_{\mathcal{M}}(\mathcal{N})^3$, respectively $S_{\mathcal{M}}(\mathcal{N})^2$.

Add to the language $\mathcal{L}$ two unary function symbols, F and G, and denote the expanded language by $\mathcal{L}'$. For fixed $p, q, r \in S_{\mathcal{M}}(\mathcal{N})$, let $\Sigma_{p,q,r}$ be the set of $\mathcal{L}'_{\mathcal{M},\mathcal{N}}$-sentences stating that F and G are $\mathcal{L}$-automorphisms, with $F(\mathcal{M}) \models p$, $G(\mathcal{M}) \models q$, and $(FG)(\mathcal{M}) \models r$. That is, $\Sigma_{p,q,r} = \Sigma_0 \cup \Sigma'_{p,q,r}$ where

$$\Sigma_0 := \Big\{ \forall v \exists w_1, w_2 :\ v = F(w_1) = G(w_2) \Big\} \ \cup\ \Big\{ \forall \overline{v} :\ \varphi(\overline{v}) \leftrightarrow \varphi(F\overline{v}) \leftrightarrow \varphi(G\overline{v}) \bigm| \varphi \in \mathcal{L} \Big\}$$

$$\Sigma'_{p,q,r} := \Big\{ \varphi(F(\overline{\mathcal{M}})) \bigm| \varphi \in p \Big\} \ \cup\ \Big\{ \varphi(G(\overline{\mathcal{M}})) \bigm| \varphi \in q \Big\} \ \cup\ \Big\{ \varphi(FG(\overline{\mathcal{M}})) \bigm| \varphi \in r \Big\}.$$

Then, by construction, $(p, q, r) \in \mathcal{P}$ if and only if $\Sigma_{p,q,r}$ is satisfiable in $\mathbb{U}$, or equivalently - by saturation of $\mathbb{U}$ - if $\Sigma_{p,q,r} \cup T$ is consistent. Therefore by compactness, for any $(p, q, r) \notin \mathcal{P}$ there exist formulas $\varphi \in p$, $\chi \in q$, and $\psi \in r$ such that

$$T \cup \Sigma_0 \cup \left\{\varphi(F(\overline{\mathcal{M}})), \chi(G(\overline{\mathcal{M}})), \psi(FG(\overline{\mathcal{M}}))\right\} \text{ is inconsistent,}$$

and so for any $p', q', r' \in S_{\mathcal{M}}(\mathcal{N})$ with $\varphi \in p'$, $\chi \in q'$ and $\psi \in r'$, we get $(p', q', r') \notin P$. We conclude that $\mathcal{P}$ is closed.
To see that $\mathcal{I}$ is closed, we use a similar argument, where for fixed $p, q \in S_{\mathcal{M}}(\mathcal{N})$ we consider the set $\Sigma_{p,q}$, stating that F is an automorphism, with $F(\overline{\mathcal{M}}) \models p$ and $F^{-1}(\overline{\mathcal{M}}) \models q$. $\square$

We summarize the results in the following Corollary.

Corollary 6.20. $\mathrm{Gal}_{\mathrm{L}}(T)$ *is a compact topologized group. If* $\mathrm{Gal}_{\mathrm{L}}(T)$ *is Hausdorff, it is a topological group.*

We remark again that by Ziegler [**ziegler**], $\mathrm{Gal}_{\mathrm{L}}(T)$ is in fact always a topological group under the topology described above.
Lascar asked in [16] whether $\mathrm{Gal}_{\mathrm{L}}(T)$ is always Hausdorff. It turns out that this is not the case. Refer to [5, Section 4] for one construction of a complete, $\aleph_0$-categorcial theory T, admitting QE, that has non-Hausdorff Lascar-group.
The following chapter discussed theories whose Lascar group is Hausdorff.

Chapter 7

G-Compactness

Lascar's Reconstruction Theorem is a result about theories which Lascar calls *G-finite* [16]. That is, theories that have finite Lascar group $\mathrm{Gal_L}(T/X)$ for each finite set X, and that are *G-compact.*

As in previous chapters, we fix a theory T with QE and assume that T has saturated models of arbitrarily large regular cardinality. We fix a saturated model $\mathbb{U}$ of regular cardinality greater than $2^{|T|}$.

T is G-compact when the singleton $\{\mathrm{Autf_L}(\mathbb{U}/X)\}$ is topologcially closed in $\mathrm{Gal_L}(\mathbb{U}/X)$, for all finite sets $X \subseteq \mathbb{U}$. Topological group theory tells us that this is the case if and only if $\mathrm{Gal_L}(\mathbb{U}/X)$ is Hausdorff, for all finite X (see Remark 7.6).

Definition 7.1. We say that T is *G-compact* if $\mathrm{Gal_L}(\mathbb{U}/X)$ is Hausdorff for all finite $X \subset \mathbb{U}$.

We define in Section 7.1 the notion of Kim-Pillay-strong types, and present a proof that T is G-compact precisely when these strong types coincide with Lascar-strong types, over any finite parameter-set. As a corollary of this, we see in Section 7.2 that in the $\aleph_0$-categorical case, G-compactness is equivalent to closure of $\mathrm{Autf_L}(\mathbb{U}/X)$ under pointwise convergence (for finite all X). One direction - namely that if $\mathrm{Gal_L}(\mathbb{U}/X)$ is Hausdorff, then $\mathrm{Autf_L}(\mathbb{U}/X)$ is closed under pointwise convergence - holds even in the general case, since the projection map $\pi : \mathrm{Aut}_X(\mathbb{U}) \to \mathrm{Gal_L}(\mathbb{U}/X)$ is continuous by Remark 6.16.

Finally, we conclude Section 7.2 with a central result for the proof of Lascar's Reconstruction Theorem: For an $\aleph_0$-categorical, G-compact theory T, the Lascar group over any finite set is profinite, and - roughly speaking - the class of any $f \in \mathrm{Aut}_X(\mathbb{U})$ in $\mathrm{Gal_L}(\mathbb{U}/X)$ is determined by the actions of f on the quotients of $\mathbb{U}$ by finite, X-definable equivalence relations E. This is Theorem 7.16.

7.1 Strong Types

For a type-definable (i.e. defined by a possibly infinite conjunction of formulas) equivalence relation E on a theory T, and a cardinal κ, we will say that E is κ-*bounded* if E has at most κ-many equivalence classes in any model of T . We say that E is *finite* if it is n-bounded for some $n \in \omega$, and we say E is *bounded*, if it is κ-bounded for some cardinal $\kappa < |\mathbb{U}|$.

Recall that for any model $\mathcal{M}$ containing X, and tuples $\overline{a}, \overline{b}$ of the same length in $\mathcal{M}$, we say that $\overline{a}$ and $\overline{b}$ have the same Lascar-strong type over X (and write $\overline{a} \overset{\mathrm{L}}{\equiv}_X \overline{b}$) if there exists $f \in \mathrm{Autf_L}(\mathbb{U}/X)$ with $f(\overline{a}) = \overline{b}$. We define, following Shelah [26, Definition 2.1] the weaker notion of *Shelah-strong types*, and following [12] the intermediate notion of *Kim-Pillay-strong types* types.

Definition 7.2. For a finite set $X \subseteq \mathbb{U}$, and (possibly infinite) tuples $\overline{a}, \overline{b} \subseteq \mathbb{U}$ of length $< |\mathbb{U}|$, we say that

- $\overline{a}$ and $\overline{b}$ have the same (Shelah-)strong type over X (write $\overline{a} \overset{\mathrm{Sh}}{\equiv}_X \overline{b}$), if for every finite, X-definable equivalence relation E, we have $\mathbb{U} \models E(\overline{a}, \overline{b})$,
- $\overline{a}$ and $\overline{b}$ have the same Kim-Pillay-strong type over X (write $\overline{a} \overset{\mathrm{KP}}{\equiv}_X \overline{b}$) if for every bounded, X-type-definable equivalence relation E, we have $\mathbb{U} \models E(\overline{a}, \overline{b})$.

Moreover, we define

$$\mathrm{Autf_{Sh}}(\mathbb{U}/X) := \left\{ f \in \mathrm{Aut}(\mathbb{U}) \;\middle|\; \forall \overline{a} \in \mathbb{U}^{<\omega} : \ \overline{a} \overset{\mathrm{Sh}}{\equiv}_X f(\overline{a}) \right\}$$

and

$$\mathrm{Autf_{KP}}(\mathbb{U}/X) := \left\{ f \in \mathrm{Aut}(\mathbb{U}) \;\middle|\; \forall \overline{a} \in \mathbb{U}^{<|\mathbb{U}|} : \ \overline{a} \overset{\mathrm{KP}}{\equiv}_X f(\overline{a}) \right\}.$$

We formulate and prove some properties of these various notions of strong types for $X = \emptyset$. We only make this assumption for the purpose of readability, and it is clear that the results transfer to the general case where $X \neq \emptyset$.

Lemma 7.3. *Let $f \in \mathrm{Aut}(\mathbb{U})$, and $f' \in \mathrm{Aut}(\mathbb{U}')$ be an extension of f, where $\mathbb{U} \preccurlyeq \mathbb{U}'$. Then*

$$f \in \mathrm{Autf_{KP}}(\mathbb{U}) \iff f' \in \mathrm{Autf_{KP}}(\mathbb{U}').$$

Proof. We may assume w.l.o.g. that $\mathbb{U}'$ is saturated and larger than $\mathbb{U}$.
The implication from right to left is clear.
Suppose now that $f \in \mathrm{Autf_{KP}}(\mathbb{U})$. Let E be a bounded, type-definable equivalence relation, and let $\overline{a}'$ be a tuple in $\mathbb{U}'$ of suitable length. Since E is bounded (by some

$\kappa < \mathbb{U}$), saturation of $\mathbb{U}$ yields a tuple $\overline{a} \subset \mathbb{U}$ with $\mathbb{U}' \models E(\overline{a}, \overline{a}')$. Since E is type-definable, and therefore invariant under automorphisms, we get $\mathbb{U}' \models E(f'(\overline{a}), f'(\overline{a}'))$. Recalling also that $f \in \mathrm{Autf_{KP}}(\mathbb{U})$, we have

$$\mathbb{U}' \models E(\overline{a}', \overline{a}) \wedge E(\overline{a}, f(\overline{a})) \wedge E(f(\overline{a}), f'(\overline{a}')),$$

and so $E(\overline{a}', f'(\overline{a}'))$ by transitivity of E. We conclude that $f' \in \mathrm{Autf_{KP}}(\mathbb{U}')$. $\square$

Corollary 7.4. $\mathrm{Autf_{KP}}(\mathbb{U})$ *is an h-normal subgroup of* $\mathrm{Aut}(\mathbb{U})$*, and*

$$\mathrm{Autf_L}(\mathbb{U}) \subseteq \mathrm{Autf_{KP}}(\mathbb{U}).$$

Proof. It is clear that $\mathrm{Autf_{KP}}(\mathbb{U})$ is a normal subgroup of $\mathrm{Aut}(\mathbb{U})$, since any type-definable equivalence relation E is invariant under automorphisms, so for any $f \in \mathrm{Autf_{KP}}$, $g \in \mathrm{Aut}(\mathbb{U})$ and $\overline{a} \subset \mathbb{U}$,

$$E\big(\overline{a}, f(\overline{a})\big) \implies E\big(g(a), (gf)(a)\big) \implies E\big(g(a), (gfg^{-1})(g(a))\big).$$

Now, we apply Lemma 7.3 in an argument like that in Proposition 6.7, to see that $\mathrm{Autf_{KP}}(\mathbb{U})$ is h-normal. $\square$

We can see now that the hierarchy of strong types is as described at the beginning of this section.

Corollary 7.5. $\mathrm{Autf_L}(\mathbb{U}/X) \subseteq \mathrm{Autf_{KP}}(\mathbb{U}/X) \subseteq \mathrm{Autf_{Sh}}(\mathbb{U}/X)$.

Proof. The first inclusion is Corollary 7.4 and the second inclusion is clear. $\square$

We shall work now toward a proof of Remark 4.20 in [15], which gives a characterization of G-Compactness, using Kim-Pillay-strong types. For $\aleph_0$-categorical theories, Corollary 7.14 will show that the three notions of strong types we have defined coincide on finite tuples, and under the additional assumption of G-compactness, they coincide completely.

Following [27], we let $\Gamma_1(\mathbb{U})$ the topological closure of $\{\mathrm{Autf_L}(\mathbb{U})\}$ in $\mathrm{Gal_L}(\mathbb{U})$. Moreover, following [15], denote by $G_1(\mathbb{U})$ the preimage $\pi^{-1}(\Gamma_1(\mathbb{U})) \subseteq \mathrm{Aut}(\mathbb{U})$. $G_1(\mathbb{U})$ is topologically closed in $\mathrm{Aut}(\mathbb{U})$, since π is continuous by Remark 6.16.

Remark 7.6 ([9, page 30]). *We know from topological group theory that* $\Gamma_1(\mathbb{U}) = \overline{\{1_{\mathrm{Gal_L}(\mathbb{U})}\}}$ *is a normal subgroup of* $\mathrm{Gal_L}(\mathbb{U})$*, and therefore* $G_1(\mathbb{U})$ *is a normal subgroup of* $\mathrm{Aut}(\mathbb{U})$*. Moreover* $\mathrm{Gal_L}(\mathbb{U})$ *is Hausdorff if and only if* $\Gamma_1(\mathbb{U}) = \{1_{\mathrm{Gal_L}(\mathbb{U})}\}$*, i.e. if the singleton* $\{\mathrm{Autf_L}(\mathbb{U})\}$ *is closed in* $\mathrm{Gal_L}(\mathbb{U})$.

Proposition 7.7 ([27, Theorem 21])**.** $G_1(\mathbb{U}) = \mathrm{Autf}_{\mathrm{KP}}(\mathbb{U})$.

Proof. We show first the inclusion from left to right.
Fix a bounded, type-definable equivalence relation E on tuples of length $\kappa < |\mathbb{U}|$, and let $(\varphi_i)_{i\in I}$ be the formulas defining E, i.e.

$$\forall \overline{a}, \overline{b} \in \mathbb{U}^\kappa : \big[\, E(\overline{a}, \overline{b}) \iff \forall i \in I :\ \mathbb{U} \models \varphi_i(\overline{a}, \overline{b})\big].$$

Now, fix a tuple $\overline{a} \in \mathbb{U}^\kappa$, and let Γ_E^a be the set of all classes $f \cdot \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}) \in \mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ where $E(\overline{a}, f(\overline{a}))$, i.e.

$$\Gamma_E^a := \big\{G \in \mathrm{Gal}_{\mathrm{L}}(\mathbb{U}) \mid \forall f \in \pi^{-1}(G) \forall i \in I :\ \mathbb{U} \models \varphi_i(\overline{a}, f(\overline{a}))\big\}.$$

If we can show that $\Gamma_1(\mathbb{U}) \subseteq \Gamma_E^a$, then we have shown the inclusion.
Clearly, $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U}) \in \Gamma_E^a$, by Corollary 7.5. Moreover, Γ_E^a is closed under composition, since

$$E\big(\overline{a}, f(\overline{a})\big) \wedge E\big(\overline{a}, g(\overline{a})\big) \implies E\big(g(\overline{a}), gf(\overline{a})\big) \wedge E\big(\overline{a}, g(\overline{a})\big) \implies E\big(\overline{a}, gf(\overline{a})\big).$$

We conclude that $\Gamma_E^a \leq \mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ is a subgroup. Now, by definition of $\Gamma_1(\mathbb{U})$, it is enough to show that Γ_E^a is topologically closed.

<u>Claim:</u> Γ_E^a is closed.
Since $|\overline{a}| < |\mathbb{U}|$, we can find a small model $\mathcal{M} \preccurlyeq \mathbb{U}$ with $\overline{a} \subseteq \mathcal{M}$. Now, with $\tau : S_{\mathcal{M}}(\mathcal{M}) \to \mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ defined as in Section 6.3, we see that

$$\begin{aligned}\tau^{-1}\big(\Gamma_E^a\big) &= \big\{\, \mathrm{tp}\,\big(f(\overline{\mathcal{M}})/\mathcal{M}\big) \;\big|\; f \in \mathrm{Aut}(\mathbb{U}) \text{ s.t. } E\big(\overline{a}, f(\overline{a})\big)\big\} \\ &= \big\{p \in S_{\mathcal{M}}(\mathcal{M}) \mid \forall i \in I :\ \varphi_i(\overline{v}, \overline{a}) \in p\big\}\end{aligned}$$

where $\overline{v}$ is the tuple of variables with the same indices as $\overline{a}$ in $\overline{\mathcal{M}}$. It follows that $\tau^{-1}(\Gamma_E^a)$ is closed as an intersection of clopen sets in $S_{\mathcal{M}}(\mathcal{M})$, so by definition, Γ_E^a is closed in $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$.

We show now the inclusion from right to left. To do so, we show that for every $n \in \omega$, the equivalence relation

$$E^n(\overline{a}, \overline{b}) :\iff \exists f \in G_1(\mathbb{U}) :\ f(\overline{a}) = \overline{b}$$

on $\mathbb{U}^n$ is type-definable and bounded. Then, $\mathrm{Autf}_{\mathrm{KP}}(\mathbb{U}) \subseteq G_1(\mathbb{U})$ will follow by definition of $\mathrm{Autf}_{\mathrm{KP}}(\mathbb{U})$, and topological closure of $G_1(\mathbb{U})$.
Note first that $E^n(\overline{a}, \overline{b})$ only depends on the type $\mathrm{tp}(\overline{a}\overline{b})$: If $\mathrm{tp}(\overline{a}\overline{b}) = \mathrm{tp}(\overline{a}'\overline{b}')$ and

$f(\overline{a}) = \overline{b}$ for some $f \in G_1(\mathbb{U})$, then find by homogeneity of $\mathbb{U}$, $g \in \operatorname{Aut}(\mathbb{U})$ with $g(\overline{a}\overline{b}) = \overline{a}'\overline{b}'$. Then

$$(g \circ f \circ g^{-1})(\overline{a}') = g(f(\overline{a})) = g(\overline{b}) = \overline{b}',$$

and since $G_1(\mathbb{U})$ is normal by 7.6 , we get $E^n(\overline{a}', \overline{b}')$.
Thus, if we define P to be the set of all $2n$-types of $\mathbb{U}$ which are realized by some pair in E^n, we get

$$\forall \overline{a}, \overline{b} \in \mathbb{U}^n : \ \neg E^n(\overline{a}, \overline{b}) \iff \bigvee_{p \in P^c} \bigwedge_{\varphi \in p} \varphi(\overline{a}, \overline{b}).$$

Fix two small models $\mathcal{M}, \mathcal{N} \preccurlyeq \mathbb{U}$. Since $\Gamma_1(\mathbb{U})$ is closed, its preimage under τ is closed in $S_{\mathcal{M}}(\mathcal{N})$. That is, there exists a set Φ of $\mathcal{L}_{\mathcal{M},\mathcal{N}}$-formulas such that

$$\forall f \in \operatorname{Aut}(\mathbb{U}) : \ f \in G_1(\mathbb{U}) \iff \Phi \subseteq \operatorname{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big).$$

Now, let $\mathcal{L}'$ be the expansion of $\mathcal{L}$ by a unary function symbol F and an n-tuple of constants $\overline{c}$. For a type $p \in \mathcal{L}(2n)$, denote by Σ_p the set of $\mathcal{L}'_{\mathcal{M},\mathcal{N}}$-formulas, stating that

- F is an $\mathcal{L}$-isomorphism,
- $\Phi \subseteq \operatorname{tp}\big(F(\overline{\mathcal{M}})/\mathcal{N}\big)$, and
- $\operatorname{tp}\big(F(\overline{c})\overline{c}\big) = p$.

By our above observations, Σ_p is consistent with T precisely when $p \in P$. By compactness, it follows that for any $p \notin P$, there exists $\varphi_p \in p$ such that for any other type $p' \in S_{2n}(\mathbb{U})$ with $\varphi_p \in p'$, we get $p' \notin P$. We conclude that

$$\forall \overline{a}, \overline{b} : \neg E^n(\overline{a}, \overline{b}) \iff \bigvee_{p \in P^c} \varphi_p(\overline{a}, \overline{b}),$$

and therefore E^n type-definable.
Finally, E^n is bounded, since

$$\big|\operatorname{Aut}(\mathbb{U})/G_1(\mathbb{U})\big| \leq \big|\operatorname{Aut}(\mathbb{U})/\operatorname{Autf}_{\mathrm{L}}(\mathbb{U})\big| \overset{6.15}{\leq} 2^{|T|}.$$

$\square$

By the above Proposition, we see that $\operatorname{Gal}_{\mathrm{L}}(\mathbb{U})$ is Hausdorff, i.e. $\Gamma_1(\mathbb{U}) = \{\operatorname{Autf}_{\mathrm{L}}(\mathbb{U})\}$, exactly when Lascar-strong types coincide with Kim-Pillay-strong types.

Proposition 7.8 ([15, [Remark 4.20]). *The following are equivalent:*

(i) $\operatorname{Gal}_{\mathrm{L}}(\mathbb{U})$ *is Hausdorff.*

(ii) $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U}) = \mathrm{Autf}_{\mathrm{KP}}(\mathbb{U})$ $\big(= G_1(\mathbb{U})\big)$.

(iii) $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$ *is topologically closed, and for all finite tuples* $\overline{a}, \overline{b} \in \mathbb{U}^{<\omega}$,

$$\overline{a} \overset{\mathrm{KP}}{\equiv} \overline{b} \iff \overline{a} \overset{\mathrm{L}}{\equiv} \overline{b}.$$

Proof. Recall that by Remark 7.6, $G_1(\mathbb{U}) \trianglelefteq \mathrm{Aut}(\mathbb{U})$.
(i) $\iff$ *(ii)* Follows from Remark 7.6 and Proposition 7.7.
(ii) $\implies$ *(iii)*: $G_1(\mathbb{U})$ is closed as a continuous preimage of $\Gamma_1(\mathbb{U})$, and $\mathrm{Autf}_{\mathrm{KP}}(\mathbb{U}) = G_1(\mathbb{U})$ by Proposition 7.7.
(iii) $\implies$ *(ii)*: $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U}) \subseteq \mathrm{Autf}_{\mathrm{KP}}(\mathbb{U})$ holds by Corollary 7.5. Now, let $f \in \mathrm{Autf}_{\mathrm{KP}}(\mathbb{U})$. Then for any $\overline{a} \in \mathbb{U}^{<\omega}$, by *(iii)*, we get $\overline{a} \overset{\mathrm{L}}{\equiv} f(\overline{a})$. Since $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$ is closed by assumption, we conclude that $f \in \mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$. □

From this, we immediately get the following characterization of G-compactness.

Corollary 7.9. *T is G-compact if and only if for all finite $X \subseteq \mathbb{U}$, $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U}/X)$ is closed, and $\overset{\mathrm{L}}{\equiv}_X$ agrees with $\overset{\mathrm{KP}}{\equiv}_X$ on finite tuples.*

We end this section with some general statements about Lascar-strong types, leading up to the $\aleph_0$-categorical case in Section 7.2.
For a finite set $X \subseteq \mathbb{U}$ and $n \in \omega$, define $R^n_X(\overline{a}, \overline{b})$ to be Lascar-equivalence on tuples of length n, i.e.

$$R^n_X(\overline{a}, \overline{b}) \iff \overline{a} \overset{\mathrm{L}}{\equiv}_X \overline{b}$$

for $\overline{a}, \overline{b} \in \mathbb{U}^n$. This clearly forms an equivalence relation.

Lemma 7.10. *R^n_X has at most $2^{|T|}$-many classes on any model.*

Proof. The number of R^n_X-classes is bounded by $\big|S_n(T/X)\big| \cdot \big|\,\mathrm{Gal}_{\mathrm{L}}(T/X)\big|$. We know that $\big|S_n(T/X)\big| \leq 2^{|T|}$, and by Remark 6.15, $\big|\,\mathrm{Gal}_{\mathrm{L}}(T/X)\big| \leq 2^{|T|}$. □

Lemma 7.11. *Whether $R^n_X(\overline{a}, \overline{b})$ holds depends only on the type* $\mathrm{tp}(\overline{a}\overline{b}/X)$.

Proof. This follows by an argument like in Proposition 7.7, using the fact that $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$ is normal. □

7.2 G-Compactness for $\aleph_0$-categorical Theories

We will consider in this section the case where T is $\aleph_0$-categorcial. As usual $\mathbb{U}$ denotes a saturated model of regular cardinality $> 2^{|T|}$ (in this case $|\mathbb{U}| > 2^{\aleph_0}$). X denotes a finite subset of $\mathbb{U}$. Note that the theory T_X with additional constants for elements of X is still $\aleph_0$-categorical, so again, we may assume in our proofs that $X = \emptyset$.

Lemma 7.11 tells us that for fixed $n \in \omega$, there exists a family of types $(p_i)_{i \in I}$ in $S_n(T)$ such that

$$R_X^n(\bar{a}, \bar{b}) \iff \operatorname{tp}(\bar{a}\bar{b}/X) \in \{p_i \mid i \in I\} \iff \mathbb{U} \models \bigvee_{i \in I} \bigwedge_{\varphi \in p_i} \varphi(\bar{a}, \bar{b}). \tag{7.1}$$

If additionally T is $\aleph_0$-categorical, the Stone Space $S_n(T)$ is finite by the Ryll-Nardzewski Theorem, so $(p_i)_{i \in I}$ is finite, and R_X^n is X-type-definable.
Again by the Ryll-Nardzewski Theorem, we know that all types in $S_n(T)$ are isolated. In particular, any X-type-definable relation on $\mathbb{U}^n$ is definable. We conclude that R_X^n is in fact X-definable. Finally, we observe that R_X^n is also finite.

Lemma 7.12. *R_X^n is a finite, X-definable equivalence relation.*

Proof. For simplicity, assume that $n = 1$.
We have already seen that R_X^1 is an X-definable equivalence relation, so there exists a formula $\varphi(\bar{v}, \bar{w}) \in \mathcal{L}_X$ which defines R_X^1. Moreover, by Lemma 7.10, R_X^1 is bounded. That is, there exists a cardinal $\kappa < |\mathbb{U}|$ such that R_X^1 has at most κ-many equivalence classes in any model of T_X. Therefore, the set of formulas

$$\{v_i \neq v_j \mid i < j < \kappa^+\} \cup \{\neg\varphi(v_i, v_j) \mid i < j < \kappa^+\},$$

which states that there exist κ^+-many distinct elements which are all in distinct classes of R_X^1, is inconsistent with T_X. By Compactness, there exists $k \in \omega$ such that

$$\{v_i \neq v_j \mid i < j < k\} \cup \{\neg\varphi(v_i, v_j) \mid i < j < k\}$$

is already inconsistent with T_X. That is, the number of R_X^1-classes is bounded by $k \in \omega$, so R_X^1 is finite. $\square$

Given Lemma 7.12, it is easy to see that Lascar-strong types and strong types coincide on finite tuples in $\aleph_0$-categorical theories. Together with Proposition 7.8, this yields the following Corollary.

Corollary 7.13. *If T is $\aleph_0$-categorical, then the following are equivalent.*

(i) $\mathrm{Gal_L}(\mathbb{U})$ *is Hausdorff.*

(ii) $\mathrm{Autf_L}(\mathbb{U}) = \mathrm{Autf_{KP}}(\mathbb{U}) = \mathrm{Autf_{Sh}}(\mathbb{U})$.

(iii) $\mathrm{Autf_L}(\mathbb{U})$ *is closed.*

Proof. By Lemma 7.12 and Corollary 7.5, we have

$$\overline{a} \overset{\mathrm{Sh}}{\equiv} \overline{b} \iff \overline{a} \overset{\mathrm{L}}{\equiv} \overline{b},$$

for any finite tuples $\overline{a}, \overline{b} \in \mathbb{U}^n$, and if $\mathrm{Autf_L}(\mathbb{U})$ is closed, it follows that $\mathrm{Autf_L}(\mathbb{U}) = \mathrm{Autf_{Sh}}(\mathbb{U})$. Now, the statement follows immediately from Proposition 7.8. □

As a result, we see that an $\aleph_0$-categorical theory T is G-compact (7.1) precisely when $\mathrm{Autf_L}(\mathbb{U}/X)$ is closed under pointwise convergence.

Corollary 7.14. *If* T *is* $\aleph_0$*-categorical, then* T *is G-compact if and only if* $\mathrm{Autf_L}(\mathbb{U}/X)$ *is closed in* $\mathrm{Aut}_X(\mathbb{U})$ *for all finite* $X \subseteq \mathbb{U}$.

We will assume for the rest of this section that T is G-compact. Then by Propositions 6.18 and 6.19, we know that $\mathrm{Gal_L}(T)$ is a compact, Hausdorff topological group. Given this, topological group theory (see for example [9, Propositions 6 and 14]) tells us the following.

Fact 7.15. (i) Any open subgroup of $\mathrm{Gal_L}(T)$ is closed and any closed subgroup has finite index.

(ii) $\mathrm{Gal_L}(T)$ is totally disconnected (that is, its connected components are all singletons) if and only if the intersection of all its (cl)open normal subgroups is trivial.

With this in mind, we prove the central theorem of this chapter. For $n \in \omega$ and finite $X \subset \mathbb{U}$, we shall denote by E_n^X the intersection of all finite, X-definable equivalence relations on $\mathbb{U}^n$, which - by $\aleph_0$-categoricity - is finite and X-definable itself. Moreover, we write

$$\mathcal{H}_n^X := \Big\{ F \in \mathrm{Gal_L}(\mathbb{U}/X) \;\Big|\; \forall \mathbb{U}' \succcurlyeq \mathbb{U} \text{ satd. } \forall f' \in \mathrm{Aut}_X(\mathbb{U}') \text{ s.t. } f'|_{\mathbb{U}} \in F : E_n\big(\overline{a}, f'(\overline{a})\big) \text{ for all } \overline{a} \in \mathbb{U}'^n \Big\}.$$

Notice that since $E\big((\overline{a}, a_{n+1}), (\overline{b}, b_{n+1})\big) :\equiv E_n^X(\overline{a}, \overline{b})$ is an X-definable, finite equivalence relation on $\mathbb{U}^{n+1}$, we get $\mathcal{H}_n^X \supseteq \mathcal{H}_{n+1}^X$ for all $n \in \omega$.

Theorem 7.16 (Proposition 510 in [16]). *If T is $\aleph_0$-categorical and G-compact, then for any finite $X \subset \mathbb{U}$, $\mathrm{Gal_L}(T/X)$ is totally disconnected. Moreover, for each $n \in \omega$, $\mathcal{H}_n^X$ is a clopen normal subgroup of $\mathrm{Gal_L}(\mathbb{U})$, and*

$$\bigcap_{n=1}^{\infty} \mathcal{H}_n^X = \{\mathrm{Autf_L}(\mathbb{U}/X)\}.$$

Proof. Assume w.l.o.g. that $X = \emptyset$, and write $\mathcal{H}_n$ for $\mathcal{H}_n^{\emptyset}$.
By $\aleph_0$-categoricity, for any $n \in \omega$, E_n is definable and finite.
We begin by showing that the sets $\mathcal{H}_n$ are open normal subgroups. For simplicity we assume $n = 1$, and write $E := E_1^X$.The proof is analogous for higher $n \in \omega$.
By Corollary 7.13, we have

$$\mathrm{Autf_L}(\mathbb{U}) = \mathrm{Autf_{Sh}}(\mathbb{U}). \tag{7.2}$$

<u>Claim 1:</u> $\mathcal{H}_1$ is a normal subgroup.
By (7.2), $1_{\mathrm{Gal_L}(\mathbb{U})} = \mathrm{Autf_L}(\mathbb{U}) \in \mathcal{H}_1$.
Now let $F, G \in \mathcal{H}_1$, and let $fg \in FG$. Moreover, let $\mathbb{U}' \succcurlyeq \mathbb{U}$ be a saturated supermodel, and $h' \in \mathrm{Aut}(\mathbb{U}')$ be an extension of fg. Then find (by homogeneity of $\mathbb{U}'$) automorphisms $f', g' \in \mathrm{Aut}(\mathbb{U}')$ extending f and g, respectively. Then $h'|_{\mathbb{U}} = (f'g')|_{\mathbb{U}}$, so

$$h' \in (f'g') \cdot \mathrm{Autf_L}(\mathbb{U}') \overset{7.13}{=} (f'g') \cdot \mathrm{Autf_{Sh}}(\mathbb{U}'),$$

where the second equality holds since Corollary 7.13 is independent of the choice of large saturated model, and therefore holds for $\mathbb{U}'$. Now for any $a \in \mathbb{U}'$, we get

$$E\big(a, g'(a)\big) \wedge E\big(g'(a), (f'g')(a)\big) \implies E\big(a, (f'g')(a)\big)$$

since $G, H \in \mathcal{H}_1$ and E is transitive. Now, since $h \in (f'g')\,\mathrm{Autf_{Sh}}(\mathbb{U}')$, we get $E\big(h'(a), (f'g')(a)\big)$ and again by transitivity, $E\big(h'(a), a\big)$. Since h' was an arbitrary extension of an arbitrary element of FG, we conclude that $FG \in \mathcal{H}_1$, and $\mathcal{H}_1$ is a subgroup.
Now, let $F \in \mathcal{H}_1$ and $H \in \mathrm{Gal_L}(\mathbb{U})$. For a saturated model $\mathbb{U}' \succcurlyeq \mathbb{U}$, let $f', h' \in \mathrm{Aut}(\mathbb{U}')$ with $f'|_{\mathbb{U}} \in F$ and $h'|_{\mathbb{U}} \in H$. To show that $\mathcal{H}_1$ is normal, we need to show that $(h')^{-1} f'h'$ preserves E. Fix an element $a \in \mathbb{U}'$. Then $E\big(h'(a), f'(h'a)\big)$ since $f'|_{\mathbb{U}} \in F \in \mathcal{H}_1$, and since E is definable, we can apply h'^{-1} to both sides to obtain $E\big(a, (h'^{-1}f'h')(a)\big)$. We conclude that $\mathcal{H}_1$ is a normal subgroup.

<u>Claim 2:</u> $\mathcal{H}_1$ is clopen.
By point (i) of Fact 7.15, it is enough to show that $\mathcal{H}_1$ is open in $\mathrm{Gal_L}(T)$, under the

topology described in Section 6.3. To do so, we fix two small models $\mathcal{M}, \mathcal{N} \preccurlyeq \mathbb{U}$, and show that

$$S^*_{\mathcal{M}}(\mathcal{N}) := \Big\{ \operatorname{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big) \mid f \cdot \operatorname{Autf_L}(\mathbb{U}) \in \mathcal{H}_1 \Big\}$$

is open in $S_{\mathcal{M}}(\mathcal{N})$. Note that this is well-defined, since $\operatorname{Autf_L}(\mathbb{U}) \in \mathcal{H}_1$ by Claim 1. Let $p = \operatorname{tp}\big(f(\overline{\mathcal{M}})/\mathcal{N}\big) \in S_{\mathcal{M}}(\mathcal{N})$ be in the topological closure of $S_{\mathcal{M}}(\mathcal{N}) \setminus S^*_{\mathcal{M}}(\mathcal{N})$. That is, for any finite set $\Phi \subset p$, there exists an automorphism $g_\Phi \in \operatorname{Aut}(\mathbb{U})$ with $g_\Phi \operatorname{Autf_L}(\mathbb{U}) \notin \mathcal{H}_1$, and $\Phi \subset \operatorname{tp}\big(g_\Phi(\overline{\mathcal{M}})/\mathcal{N}\big)$. If we can show that $p \notin S^*_{\mathcal{M}}(\mathcal{N})$ (or equivalently, $f \notin \mathcal{H}_1$) then we are done.

We expand the language by a unary function symbol G, and a constant c, and denote the expanded language by $\mathcal{L}'$. Then, we define $\Sigma := \Sigma_1 + \Sigma_2 + \Sigma_3$ to be the set of $\mathcal{L}'_{\mathcal{M}\mathcal{N}}$-formulas where

- Σ_1 states that G is an $\mathcal{L}-$automorphism, i.e.

$$\Sigma_1 := \Big\{ \forall \overline{v} : \ \varphi(\overline{v}) \leftrightarrow \varphi(G\overline{v}) \mid \varphi \in \mathcal{L} \Big\} \ \cup \ \big\{ \forall v \exists w : \ G(w) = v \big\},$$

- Σ_2 states that $\neg E\big(c, G(c)\big)$ (recall that E is definable),
- Σ_3 states that $\operatorname{tp}\big(G(\overline{\mathcal{M}})/\mathcal{N}\big) = p$, i.e.

$$\Sigma_3 := \Big\{ \varphi\big(G(\overline{\mathcal{M}})\big) \mid \varphi \in p \Big\}.$$

By our initial assumption, $\Sigma \cup T_{\mathcal{M}\mathcal{N}}$ is finitely satisfiable: Let $\Phi \subset p$ be finite. Recall that $\Phi \subset \operatorname{tp}\big(g_\Phi(\overline{\mathcal{M}})/\mathcal{N}\big)$, and $g_\Phi \operatorname{Autf_L}(\mathbb{U}) \notin \mathcal{H}_1$. That is, we can find a saturated model $\mathbb{U}' \succcurlyeq \mathbb{U}$, and an automorphism $g' \in \operatorname{Aut}(\mathbb{U}')$ extending $g_\Phi \cdot \operatorname{Autf_L}(\mathbb{U})$, such that the exists $a \in \mathbb{U}'^n$ with

$$\mathbb{U}' \models \neg E\big(a, g'(a)\big).$$

Now, since $g'|_{\mathbb{U}} = g_\Phi$, we get $\operatorname{tp}\big(g'(\mathcal{M})/\mathcal{N}\big) \supset \Phi$. Hence,

$$\big\langle \mathbb{U}', g', a \big\rangle \models T \cup \Sigma_1 \cup \Sigma_2 \ \cup \ \Big\{ \varphi\big(g'(\overline{\mathcal{M}})\big) \mid \varphi \in \Phi \Big\}.$$

By Compactness, we conclude there exists a model $\big\langle \mathbb{U}', g', a \big\rangle$ of $\Sigma \cup T_{\mathcal{M}\mathcal{N}}$, i.e. $g' \in \operatorname{Aut}(\mathbb{U}')$ with $\operatorname{tp}\big(g'(\overline{\mathcal{M}})/\mathcal{N}\big) = p$, and $a \in \mathbb{U}'$ with $\neg E(a, g'(a))$. Moreover, we may assume that $\mathbb{U}'$ is saturated (since $T_{\mathcal{M}\mathcal{N}}$ has arbitrarily large saturated models), and by upward Löwenheim-Skolem that $\mathbb{U}' \succcurlyeq \mathbb{U}$ with $|\mathbb{U}'| > |\mathbb{U}|$. Thus, by Lemma 2.14 and regularity of $|\mathbb{U}|$, we can find a saturated submodel $\mathcal{M}, \mathcal{N} \subseteq \mathbb{U}_1 \preccurlyeq \mathbb{U}'$ of cardinality $|\mathbb{U}|$, such that $g'|_{\mathbb{U}_1} \in \operatorname{Aut}(\mathbb{U}_1)$. We may assume w.l.o.g. that $\mathbb{U}_1 = \mathbb{U}$: Since saturated models of the same cardinality are isomorphic by Theorem 2.12, and $\mathbb{U}$ and $\mathbb{U}_1$ are saturated as models of $\mathcal{T}_{\mathcal{M},\mathcal{N}}$, we can find $h \in \mathrm{Aut}_{\mathcal{M},\mathcal{N}}(\mathbb{U}')$ with $h(\mathbb{U}) = \mathbb{U}_1$. Then $\left\langle \mathbb{U}', h^{-1}g'h, h^{-1}(\overline{a}) \right\rangle$ still models Σ.

To summarize, we have found a saturated model $\mathbb{U}' \succcurlyeq \mathbb{U}$ with an automorphism $g' \in \mathrm{Aut}(\mathbb{U}')$ such that $g'|_{\mathbb{U}} \in \mathrm{Aut}(\mathbb{U})$ and $\mathrm{tp}(g'(\overline{\mathcal{M}})/\mathcal{N}) = p = \mathrm{tp}(f(\overline{\mathcal{M}})/\mathcal{N})$, i.e.

$$g'|_{\mathbb{U}} \in f \cdot \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}).$$

Moreover, there is $a \in \mathbb{U}'$ with $\neg E(a, g'(a))$, so by definition,

$$f \cdot \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}) = g'|_{\mathbb{U}}\, \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}) \notin \mathcal{H}_1,$$

or equivalently, $p \notin S^*_{\mathcal{M}}(\mathcal{N})$. This conclues the proof that $\mathcal{H}_1$ is (cl)open.

<u>Claim:</u> $\bigcap_n \mathcal{H}_n = \{1_{\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})}\}$.
By construction of the sets $\mathcal{H}_n$, we get

$$\bigcap_{n\in\omega} \mathcal{H}_n \subseteq \{\mathrm{Autf}_{\mathrm{Sh}}(\mathbb{U})\} \overset{7.2}{=} \{\mathrm{Autf}_{\mathrm{L}}(\mathbb{U})\}.$$

The other inclusion follows from the fact that each $\mathcal{H}_n$ is a subgroup of $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U})$ by Claim 1.

In light of point (ii) of Fact 7.15, this proves that $\mathrm{Gal}_{\mathrm{L}}(T)$ is totally disconnected, and concludes the proof. □

In the Lascar Reconstruction Theorem, we are concerned only with G-finite theories, i.e. G-compact theories for which $\mathrm{Gal}_{\mathrm{L}}(T/X)$ is finite, over all finite sets X.
Note that a result by Newelski [21, Corollary 1.11] shows that G-compactness is not necessary in this definition: Any theory which has finite Lascar group over all finite sets X is already G-compact.

Definition 7.17. We say that T is *G-finite* if $\mathrm{Gal}_{\mathrm{L}}(T/X)$ is finite for all finite $X \subset \mathbb{U}$.

Under the assumption of G-finiteness and $\aleph_0$-categoricity, we get the following corollary of Theorem 7.16.

Corollary 7.18. *If T is $\aleph_0$-categorical and G-finite, then for any $\mathcal{M} \preccurlyeq \mathbb{U}$ and any finite set $X \subseteq \mathcal{M}$ there exists a finite $X \subseteq Y \subseteq \mathcal{M}$ such that*

$$\mathrm{Aut}_Y(\mathbb{U}) \subseteq \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}/X).$$

Proof. Let $X \subset M \preccurlyeq \mathbb{U}$ be finite.
Since T is G-compact, Theorem 7.16 tells us that

$$\bigcap_{n\in\omega} \mathcal{H}_n^X = \{\mathrm{Autf_L}(\mathbb{U}/X)\}.$$

Now, since $\mathrm{Gal_L}(\mathbb{U}/X)$ is finite by assumption, and $(\mathcal{H}_n^X)_{n\in\omega}$ is descending, we can find $N \in \omega$ with

$$\mathcal{H}_N^X = \{\mathrm{Autf_L}(\mathbb{U}/X)\}.$$

Recall that E_N^X denotes the intersection of the - by $\aleph_0$-categoricity finitely many - finite, X-definable equivalence relations on $\mathbb{U}^N$. Now, let $\overline{a}_1, \dots, \overline{a}_n \in \mathcal{M}^N$ be representatives of the finitely many classes in $\mathcal{M}/E_N^X$, and let $A := \{\overline{a}_1, \dots, \overline{a}_n\}$. Then, since $X \subseteq \mathcal{M} \preccurlyeq \mathbb{U}$,

$$\mathbb{U} \models \forall \overline{v}: \; E_N^X(\overline{v}, \overline{a}_1) \vee \cdots \vee E_N^X(\overline{v}, \overline{a}_n).$$

Any (saturated) extension $\mathbb{U}' \succcurlyeq \mathbb{U}$ still models this formula, so any automorphism $f \in \mathrm{Aut}_X(\mathbb{U}')$ which fixes the representatives in A satisfies

$$E_N^X\big(\overline{a}, f'(\overline{a})\big) \text{ for all } \overline{a} \in \mathbb{U}'^N.$$

Setting $Y := X \cup A \subseteq \mathcal{M}$, we get by definition of $\mathcal{H}_N^X$,

$$\mathrm{Aut}_Y(\mathbb{U}) \cdot \mathrm{Autf_L}(\mathbb{U}/X) \subseteq \mathcal{H}_N^X = \{\mathrm{Autf_L}(\mathbb{U}/X)\}$$

so $\mathrm{Aut}_Y(\mathbb{U}) \subseteq \mathrm{Autf_L}(\mathbb{U}/X)$. $\square$

Chapter 8

Lascar Independence

One of the key results of Lascar [16] states that for models $\mathcal{M}\mathcal{N}, \preccurlyeq \mathbb{U}$, which are in a certain way independent of one another, the group $\mathrm{Autf_L}(\mathbb{U})$ is generated as a group by the union of the stabilizers $\mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ and $\mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$. In the introduction of [16], Lascar uses specifically the word 'independent' to describe the relevant property of $\mathcal{M}$ and $\mathcal{N}$ giving rise to this result, but never formally treats it as an independence relation.

In this chapter, we recall the framework used by Lascar in defining this property, namely the *fundamental order*, *bounds* and *quasi-bounds*, *heirs* and *co-heirs*. From this, we define what we call *weak-* and *strong Lascar-independence*, denoted by $\mathop{\smile\!\!\!\!\!\!|}^{wL}$ and $\mathop{\smile\!\!\!\!\!\!|}^{sL}$.

We prove that $\mathop{\smile\!\!\!\!\!\!|}^{wL}$ and $\mathop{\smile\!\!\!\!\!\!|}^{sL}$ satisfy certain axioms of an independence relation, and show how they are formally related. Moreover, we give some examples of Lascar-independence in concrete theories. Finally, we reformulate the proofs of [27, Chapter 6], in particular Theorem 8.22 and Theorem 8.23, correspond to the central results of that chapter and will serve as key tools in the proof of Lascar's Reconstruction Theorem.

We fix as before a theory T which admits QE, and assume that saturated models of T of arbitrarily large regular cardinalities exist. We fix a saturated model $\mathbb{U}$ of T, of regular cardinality $> 2^{|T|}$.

We define, following [16, Chapter 6], the A-*class* of a type.

Definition 8.1. For $A \subseteq B \subseteq \mathbb{U}$ and $p \in S(\mathbb{U}/B)$, we define the A-*class* of p as

$$C_A(p) := \big\{\varphi(\overline{v}, \overline{w}) \in \mathcal{L}_A \mid \exists \overline{b} \in \mathcal{B}^n : \ \varphi(\overline{v}; \overline{b}) \in p\big\}.$$

Note we define this not just on types over models, but all types, for notational convenience.

We will now define the *fundamental order* on the set of types over models, as first introduced in Lascar and Poizat [17].

Definition 8.2. Let $X \subseteq \mathbb{U}$, $p \in S(X)$, and $\mathcal{M}_1, \mathcal{M}_2 \preccurlyeq \mathbb{U}$ with $X \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$. Moreover, let $p_1 \in S(\mathcal{M}_1)$ and $p_2 \in S(\mathcal{M}_2)$ with $p \subseteq p_1$ and $p \subseteq p_2$.

- We write $p_1 \ll_X p_2$ if $C_X(p_1) \subseteq C_X(p_2)$, and $p_1 \approx_X p_2$ if $C_X(p_1) = C_X(p_2)$.
- We say that $C_X(p_1)$ is a *bound* for p if p_1 is minimal w.r.t. $\ll_X$ in the set of all types over models which extend p.
- We say that $C_X(p_1)$ is a *quasi-bound* for p if all finite subsets of $C_X(p_1)$ are contained in a bound for p.

Based on the notion of a (quasi-)bound of a type, we define what we call *Lascar independence.*

Definition 8.3. For $\overline{a}, \overline{b} \subseteq \mathbb{U}$ and $X \subseteq \mathbb{U}$, we say that $\overline{a}$ is *weakly* (resp. *strongly*) *Lascar independent* of $\overline{b}$ over X, and write $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{wL}_X \overline{b}$ (resp. $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{sL}_X \overline{b}$) if there exists a model $\mathcal{M} \supseteq \overline{b}X$ such that $C_X(\overline{a}/\mathcal{M})$ is a quasi-bound (resp. bound) of $\mathrm{tp}(\overline{a}/X)$.

Note that if Φ is a bound for $\mathrm{tp}(\overline{a}/X)$, then there exists a model $\mathcal{M}$ containing X such that $\Phi = C_X(\overline{a}/\mathcal{M})$: We know that $\Phi = C_X(p)$ for some $p \in S(\mathcal{M}')$ where $X \subseteq \mathcal{M}' \preccurlyeq \mathbb{U}$. By saturation of $\mathbb{U}$, find $\overline{a}' \subseteq \mathbb{U}$ such that $p = \mathrm{tp}(\overline{a}'/\mathcal{M}')$. Since $\mathrm{tp}(\overline{a}'/X) = \mathrm{tp}(\overline{a}/X)$, we can find by homogeneity, $f \in \mathrm{Aut}_X(\mathbb{U})$ s.t. $f(\overline{a}') = \overline{a}$. Then $p = \mathrm{tp}(\overline{a}'/\mathcal{M}') = \mathrm{tp}(\overline{a}/f(\mathcal{M}'))$.

Remark 8.4. *For $X \subseteq \mathcal{M} \preccurlyeq \mathbb{U}$, we have $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{wL}_X \mathcal{M}$ (resp. $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{sL}_X \mathcal{M}$) if and only if $C_X(\overline{a}/\mathcal{M})$ is a quasi-bound (resp. bound) of $\mathrm{tp}(\overline{a}/X)$. One direction is immediately obvious. For the other direction, assume $C_X(\overline{a}/\mathcal{M}')$ is a (quasi-)bound for $\mathrm{tp}(\overline{a})$, for some $\mathcal{M}' \supseteq \mathcal{M}$. Then clearly, $\mathrm{tp}(\overline{a}/\mathcal{M}) \ll_X \mathrm{tp}(\overline{a}/\mathcal{M}')$, so $C_X(\overline{a}/\mathcal{M})$ is also a (quasi-)bound for $\mathrm{tp}(\overline{a})$.*
In particular, this means that for any tuple $\overline{b}$,

$$\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\limits^{wL}_{X} \overline{b} \text{ (resp. } \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\limits^{sL}_{X} \overline{b}) \iff \exists \overline{b}X \subseteq \mathcal{M} \preccurlyeq \mathbb{U} : \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\limits^{wL}_{X} \mathcal{M} \text{ (resp. } \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\limits^{sL}_{X} \mathcal{M}).$$

It is clear that weak- and strong-Lascar-independence are closely related. As the name suggests, strong-Lascar-independence implies weak-Lascar-independence. We make this concrete in the following.

Lemma 8.5. *For given $\overline{a}, X \subset \mathbb{U}$ and $X \subset \mathcal{M} \preccurlyeq \mathbb{U}$, $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{wL}_X \mathcal{M}$ if and only if*

$$\forall \varphi(\overline{v}, \overline{w}) \in C_X(\overline{a}/\mathcal{M})\ \exists X \subseteq \mathcal{M}' \preccurlyeq \mathbb{U} : \varphi(\overline{v}, \overline{w}) \in C_X(\overline{a}/\mathcal{M}') \text{ and } \overline{a} \mathop{\smile\!\!\!\!\!\!\mid}\limits^{sL}_{X} \mathcal{M}'.$$

Proof. Assume first that $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}^{wL}_X \mathcal{M}$, i.e. $C_X(\overline{a}/\mathcal{M})$ is a quasi-bound for $\mathrm{tp}(\overline{a}/X)$. Then for any $\varphi(\overline{v},\overline{w}) \in C_X(\overline{a}/\mathcal{M})$ there exists $\mathcal{M}'$ s.t. $C_X(\overline{a}/\mathcal{M}')$ contains $\varphi(\overline{v},\overline{w})$, and is a bound for $\mathrm{tp}(\overline{a}/X)$, i.e. $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}^{sL}_X \mathcal{M}'$ by the above remark.

Now suppose that for any $\varphi(\overline{v},\overline{w}) \in C_X(\overline{a}/\mathcal{M})$, there exists a bound $C_X(\overline{a}/\mathcal{M}')$ for $\mathrm{tp}(\overline{a}/X)$, containing φ. For a finite set

$$F = \{\varphi_1(\overline{v},\overline{w}_1),\ldots,\varphi_n(\overline{v},\overline{w}_n)\} \subset C_X(\overline{a}/\mathcal{M})$$

we can see that

$$\varphi(\overline{v},\overline{w}_1,\ldots,\overline{w}_n) := \varphi_1(\overline{v},\overline{w}_1) \wedge \ldots \wedge \varphi_n(\overline{v},\overline{w}_n) \in C_X(\overline{a}/\mathcal{M}),$$

so by assumption, there exists a bound $C_X(\overline{a}/\mathcal{M}')$ of $\mathrm{tp}(\overline{a}/X)$ containing F. It follows by definition that $C_X(\overline{a}/\mathcal{M})$ is a quasi-bound, so by Remark 8.4, $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}^{wL}_X \mathcal{M}$. $\square$

The following lemma summarizes some axioms of the relations $\mathop{\smile\!\!\!\!\!\!\,|}^{wL}$ and $\mathop{\smile\!\!\!\!\!\!\,|}^{sL}$, which are commonly used to characterize independence notions (see [1], [13]). We shall prove these properties here or later in this section. The investigation of additional axioms such as transitivity is left to a later work.

Lemma 8.6. *Both versions of Lascar-independence satisfy the following axioms, where $\overline{a},\overline{a}',\overline{b},\overline{b}'$ are tuples in $\mathbb{U}$, $X, X' \subseteq \mathbb{U}$, and $\mathcal{M} \preccurlyeq \mathbb{U}$ is a model.*

1. *(Invariance) If $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b}$ and $\mathrm{tp}(\overline{a}'\overline{b}'X') = \mathrm{tp}(\overline{a}\overline{b}X)$, then $\overline{a}' \mathop{\smile\!\!\!\!\!\!\,|}_{X'} \overline{b}'$.*
2. *(Right-monotonicity) If $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b}\overline{b}'$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b}$.*
3. *(Left-monotonicity over models) If $\overline{a}\overline{a}' \mathop{\smile\!\!\!\!\!\!\,|}_{\mathcal{M}} \overline{b}$ then $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_{\mathcal{M}} \overline{b}$.*
4. *(Weak base-monotonicity) If $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b}$ then $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \mathrm{acl}(\overline{b})$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{acl}(X)} \overline{b}$.*
5. *(Normality) $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b} \iff \overline{a}X \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b} \iff \overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b}X$.*
6. *(Existence) For $X \subseteq Y \subseteq \mathbb{U}$, there exists $Y' \subseteq \mathbb{U}$ with $\mathrm{tp}(XY) = \mathrm{tp}(XY')$ and $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}^{sL}_X Y'$.*
7. *(Left-finite character) If $\overline{a}' \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b}$ for all finite $a' \subseteq a$, then $\overline{a} \mathop{\smile\!\!\!\!\!\!\,|}_X \overline{b}$.*

Weak-Lascar-independence, additionally, satisfies the following property.

8. *(Right-strong finite character) If $\overline{a} \not\mathop{\smile\!\!\!\!\!\!\,|}^{wL}_X \mathcal{M}$ for a model $\mathcal{M} \supseteq X$, then there exists $\varphi \in \mathrm{tp}(\overline{\mathcal{M}}/\overline{a}X)$ such that for any $\overline{b}' \subseteq \mathbb{U}$ with $\varphi \in \mathrm{tp}(\overline{b}'/\overline{a}X)$, $\overline{a} \not\mathop{\smile\!\!\!\!\!\!\,|}^{wL}_X \overline{b}'$.*

Proof. We shall give arguments which apply to both weak- and strong-Lascar-independence.

1. Let $\overline{b}X \subseteq \mathcal{M} \preccurlyeq \mathbb{U}$ be s.t. $C_X(\overline{a}/\mathcal{M})$ is a (quasi-)bound for $\mathrm{tp}(\overline{a}/X)$. Then, let $f \in \mathrm{Aut}(\mathbb{U})$ be s.t. $f(\overline{a}\overline{b}X) = \overline{a}'\overline{b}'X'$. Setting $\mathcal{M}' := f(\mathcal{M})$, we see that $\overline{b}' \subseteq \mathcal{M}'$ and since
$$\mathrm{tp}(\overline{a}'X'\mathcal{M}') = \mathrm{tp}\left(f(\overline{a}X\mathcal{M})\right) = \mathrm{tp}(\overline{a}X\mathcal{M}),$$
$C_{X'}(\overline{a}'/\mathcal{M}')$ is a (quasi-)bound for $\mathrm{tp}(\overline{a}'/X')$.

2. This is clear by definition.

3. This follows from Proposition 8.14, and the fact that if every formula in $\mathrm{tp}(\mathcal{M}\overline{b}/aa')$ is realized in $\mathcal{M}$, then in particular, every formula in $\mathrm{tp}(\mathcal{M}\overline{b}/a)$ is realized in $\mathcal{M}$.

4. The first part is clear.

   Suppose $\overline{a} \mathop{\smile\!\!\!\!\!\!\!\;|}^{sL}_{X} \overline{b}$. Let $\mathcal{M} \supseteq X\overline{b}$ be a model s.t. $C_X(\overline{a}/\mathcal{M})$ is a bound for $\mathrm{tp}(\overline{a}/X)$. Then $\mathrm{acl}(X) \subseteq \mathcal{M}$. Moreover, assume that $\mathcal{M}' \supseteq \overline{b} \cup \mathrm{acl}(X)$ is another model, with
$$C_{\mathrm{acl}(X)}(\overline{a}/\mathcal{M}') \subseteq C_{\mathrm{acl}(X)}(\overline{a}/\mathcal{M}). \tag{8.1}$$
   We claim then that in fact, equality holds. Let $\psi \in \mathcal{L}_X$ be a formula (w.l.o.g. in one free variable) with finitely many solutions $y_1, \ldots, y_m$ in $\mathbb{U}$, and let $\varphi(\overline{u}, \overline{v}, \overline{w}) \in \mathcal{L}_X$. Since $\overline{a} \mathop{\smile\!\!\!\!\!\!\!\;|}^{sL}_{X} \mathcal{M}$, we have $C_X(\overline{a}/\mathcal{M}) = C_X(\overline{a}/\mathcal{M}')$, and therefore, there are equally as many $i = 1, \ldots, m$ for which $\varphi(\overline{a}, y_i, \overline{v})$ is realized in $\mathcal{M}$, as those for which this is realized in $\mathcal{M}'$. Now, by (8.1), any $\varphi(\overline{a}, y_i, \overline{v})$ which is realized in $\mathcal{M}'$ is realized in $\mathcal{M}$. We conclude that for all $i = 1, \ldots, m$,
$$\varphi(\overline{a}, y_i, \overline{w}) \in C_{\mathrm{acl}(X)}(\overline{a}/\mathcal{M}) \iff \varphi(\overline{a}, y_i, \overline{w}) \in C_{\mathrm{acl}(X)}(\overline{a}/\mathcal{M}').$$
   The result transfers to $\mathop{\smile\!\!\!\!\!\!\!\;|}^{wL}$is similar: We can assume w.l.o.g. that $\overline{b} = \mathcal{M}$ is a model. If $\overline{a} \mathop{\smile\!\!\!\!\!\!\!\;|}^{wL}_{X} \mathcal{M}$, then Lemma 8.5 and weak base-monotonicity of $\mathop{\smile\!\!\!\!\!\!\!\;|}^{sL}$ yields
$$\forall \varphi \in C_X(\overline{a}/\mathcal{M}) \;\exists\, \mathrm{acl}(X) \subseteq \mathcal{M}' : \; \varphi \in C_X(\overline{a}/\mathcal{M}') \text{ and } \overline{a} \mathop{\smile\!\!\!\!\!\!\!\;|}^{sL}_{\mathrm{acl}(X)} \mathcal{M}'. \tag{8.2}$$
   Let now y be one of finitely many solutions $y_1, \ldots, y_m$ of a formula $\psi \in \mathcal{L}_X$, and $\varphi \in \mathcal{L}_X$ such that $\mathcal{M}$ realizes $\varphi(\overline{a}, \overline{y}, \overline{v})$. We may assume that $\mathrm{tp}(y/\overline{a}X) = \mathrm{tp}(y_i/\overline{a}X)$ for all $i = 1, \ldots, m$, otherwise we can expand ψ or φ accordingly. Now by (8.2), we can find a model $\mathcal{M}' \supseteq X$ and $\overline{y}', \overline{x}$ in $\mathcal{M}'$ such that $\psi(\overline{y}') \wedge \varphi(\overline{a}, \overline{y}', \overline{x})$

holds. Since $\mathrm{tp}(\overline{y}/\overline{a}X) = \mathrm{tp}(\overline{y}'/\overline{a}X)$, we can find $f \in \mathrm{Aut}_{aX}(\mathbb{U})$ with $f(\overline{y}') = \overline{y}$. Then by invariance, $\overline{a} \mathop{\smile\!\!\!\!\!|}^{sL}_{X} f(\mathcal{M}')$, and therefore $\overline{a} \mathop{\smile\!\!\!\!\!|}^{sL}_{\mathrm{acl}(X)} f(\mathcal{M}')$, and $f(\mathcal{M}')$ realizes $\varphi(\overline{a}, \overline{y}, \overline{v})$. By Lemma 8.5, we are done.

5. The first equivalence follows from the fact that for any model $\mathcal{M}$, $C_X(\overline{a}/\mathcal{M})$ and $C_X(\overline{a}X/\mathcal{M})$ are equal (up to formally identifying an expression $\varphi(\overline{x})$ where $\varphi(\overline{v}) \in \mathcal{L}$, as a formula in $\mathcal{L}_X$). The second equivalence follows immediately by definition.

6. Find by Proposition 8.16 a model $X \subseteq \mathcal{M} \preccurlyeq \mathbb{U}$ with $\overline{a} \mathop{\smile\!\!\!\!\!|}^{sL}_{X} \mathcal{M}$. Now, let $(c_y)_{y\in Y}$ be new constants, P a new unary relation symbol, and define Σ to be the set of $\mathcal{L}_{\overline{a}X} \cup (c_y)_{y\in Y} \cup P$-sentences stating that P defines an $\mathcal{L}$-model containing $(c_y)_{y\in Y}$ and X, and that $\mathrm{tp}(X(c_y)_{y\in Y}) = \mathrm{tp}(XY)$, together with

$$\left\{\nexists \overline{v} :\ P(\overline{v}) \wedge \varphi(\overline{a}, \overline{v}) \;\middle|\; \varphi(\overline{u}, \overline{v}) \notin C_X(\overline{a}/\mathcal{M}\right\}.$$

By assumption, Σ is finitely satisfiable in $\mathbb{U}$. By compactness, we can find a model of $T \cup \Sigma$ and by construction of Σ, this means that we can find a model $\mathcal{M}' \supseteq X$ of T (w.l.o.g. $\mathcal{M}' \preccurlyeq \mathbb{U}$ by downward L-S and saturation) and $Y' \subseteq \mathcal{M}'$ with $\mathrm{tp}(XY) = \mathrm{tp}(XY')$, such that $C_X(\overline{a}/\mathcal{M}') \subseteq C_X(\overline{a}/\mathcal{M})$. Since $\overline{a} \mathop{\smile\!\!\!\!\!|}^{sL}_{X} \mathcal{M}$, we conclude $\overline{a} \mathop{\smile\!\!\!\!\!|}^{sL}_{X} \mathcal{M}'$ and therefore $\overline{a} \mathop{\smile\!\!\!\!\!|}^{sL}_{X} Y'$.

7. By assumption, find for each finite $a' \subseteq a$, a model $\mathcal{M}_{a'} \supseteq \overline{b}X$ such that $C_X(\overline{a}'/\mathcal{M}_{a'})$ is a bound for $\mathrm{tp}(\overline{a}'/X)$. Now, let $\mathcal{L}'$ be the extension of $\mathcal{L}$ by a unary relation symbol P, as well as constants for X, $\overline{a}$ and $\overline{b}$, and define the set Φ of $\mathcal{L}'$-sentences as follows.

$$\Phi := \left\{\nexists \overline{v} :\ P(\overline{v}) \wedge \varphi(\overline{a}', \overline{v}) \;\middle|\; \varphi \in \mathcal{L}_X, \overline{a}' \subseteq \overline{a}, \varphi(\overline{u}, \overline{v}) \notin C_X(\overline{a}'/\mathcal{M}_{a'})\right\}.$$

Moreover, let Σ be the set of $\mathcal{L}'$-sentences stating that P is the universe of a model of T, containing $\overline{b}$ and X. Then $\Phi \cup \Sigma \cup T$ is finitely satisfiable, by assumption. By Compactness, we can find a model $\mathbb{U}'$ of $\Phi \cup \Sigma \cup T$. Then we may assume w.l.o.g. that the model $\mathcal{M}$ defined by M in $\mathbb{U}'$ is a submodel of $\mathbb{U}$, and $\mathcal{M}$ witnesses $\overline{a} \mathop{\smile\!\!\!\!\!|}^{sL}_{X} \overline{b}$.

8. If $\overline{a} \not\mathop{\smile\!\!\!\!\!|}^{wL}_{X} \mathcal{M}$, then by Lemma 8.5, there exists a formula $\varphi(\overline{v}, \overline{w}) \in C_X(\overline{a}/\mathcal{M})$ such that for any other model $\mathcal{M}'$ containing X, if $\varphi(\overline{v}, \overline{w}) \in C_X(\overline{a}/\mathcal{M}')$, then $\overline{a} \not\mathop{\smile\!\!\!\!\!|}^{wL}_{X} \mathcal{M}'$. In particular, by adjusting the indices of the variables $\overline{v}$ accordingly, we get $\varphi(\overline{v}; \overline{a}) \in \mathrm{tp}(\overline{\mathcal{M}}/\overline{a}X)$, and for any $\overline{b}' \subseteq \mathbb{U}$ with $\varphi(\overline{v}; \overline{a}) \in \mathrm{tp}(\overline{b}'/\overline{a}X)$, we get $\overline{a} \not\mathop{\smile\!\!\!\!\!|}^{wL}_{X} \overline{b}'$.

$\square$

Let us now look at some concrete examples of Lascar-independence in well-known $\aleph_0$-categorical theories.

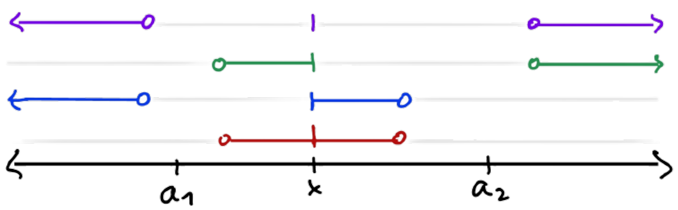
Example 8.7. Consider DLO, the theory of dense linear orders without endpoints. DLO has QE, and has unique countable model $\langle \mathbb{Q}, < \rangle$. For a finite ascending tuple $\overline{a}$ in some large sufficiently saturated model $\mathbb{U}$, and a countable model $\mathcal{M} \preccurlyeq \mathbb{U}$, we have $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{wL} \mathcal{M}$ (or equivalently $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL} \mathcal{M}$) if and only if $\mathcal{M}$ lies between two successive elements of $\overline{a}$, or completely outside of the interval $[a_1, a_m]$, i.e. if

$$\forall b \in \mathcal{M} : \ b < a_1 \ \vee \ \forall b \in \mathcal{M} : \ a_m < b \ \vee \ \bigvee_{i=1}^{n} \forall b \in \mathcal{M} : \ a_i < b < a_{i+1}.$$

Example 8.8. Still working in DLO, we fix now a parameter $x \in \mathbb{U}$, and again, elements $a_1 < a_2 < \cdots < a_m$. Weak- and strong-Lascar-independence of $\overline{a}$ over x will coincide. We have $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL}_{x} \mathcal{M}$ if and only if for some $0 \leq j < k \leq m$ (setting $a_0 := -\infty$ and $a_{m+1} := \infty$)

$$\forall b \in \mathcal{M} : \ \big[b < x \Rightarrow a_j < b < a_{j+1}\big] \ \wedge \ \big[x < b \Rightarrow a_k < b < a_{k+1}\big].$$

The case where $m = 2$ and $a_1 < x < a_2$ is visualized below.



This example shows neither $\mathop{\smile\!\!\!\!\!\!|}^{sL}$ nor $\mathop{\smile\!\!\!\!\!\!|}^{wL}$ is symmetric: Let $a < b < c < d$ be any elements of $\mathbb{U}$. Then $\{a, d\} \mathop{\smile\!\!\!\!\!\!|}^{sL} \{b, c\}$ since we can find a model containing b and c within the interval (a, d). However, $\{b, c\} \not\mathop{\smile\!\!\!\!\!\!|}^{wL} \{a, d\}$, since any model $\mathcal{M}$ containing both a and d realizes the formulas $v < a$ and $d < v$, but we can easily find another model lying completely to the left of a, or completely to the right of d.

Example 8.9. Let us look now at DLO$[-1]$, the theory of dense circular orders, whose unique countable model is $\langle \mathbb{Q}, C(u, v, w) \rangle$ where

$$C(x, y, z) \iff x < y < z \ \vee \ z < x < y \ \vee \ y < z < x.$$

Fix again some large countably saturated model $\mathbb{U}$. For a single element $a \in \mathbb{U}$, any model $\mathcal{M} \subseteq \mathbb{U}$ containing at least two elements realizes all of the formulas $C(a, v, w)$, $C(u, a, w)$, and $C(u, v, a)$. Therefore, $a \mathop{\smile\!\!\!\!\!\mid}^{sL} \overline{b}$ whenever $a \notin \overline{b}$.
Let now $a_1, a_2 \in \mathbb{U}$ be two distinct elements. Clearly, any model $\mathcal{M} \preccurlyeq \mathbb{U}$ realizes at least one of the formulas $C(a_1, v, a_2)$ or $C(a_2, v, a_1)$, but not necessarily both. We see therefore that

$$(a_1, a_2) \mathop{\smile\!\!\!\!\!\mid}^{wL} \overline{b} \iff a_1, a_2 \notin \overline{b} \ \wedge \ \left[\nexists x, y \in \overline{b} : \ C(a_1, x, a_2) \wedge C(a_2, y, a_1)\right].$$

In other words, if we picture $\mathbb{U}$ as a circle, and a_1 and a_2 as poles, then they are independent of any set that lies entirely on one side of the axis from a_1 to a_2.
Finally, let $\overline{a} \in \mathbb{U}^n$ be a finite tuple of arbitrary length $n \in \omega$. Again, if we picture $a_1, \ldots, a_n$ as points on a circle, then $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{sL} \overline{b}$ for any set $\overline{b}$ which lies entirely between two consecutive of these points.

Example 8.10. Consider the theory of the random graph, i.e. the model completion of the theory of a symmetric, irreflexive binary relation $\sim$, and let $\mathbb{U}$ be a large saturated model. For a single element $a \in \mathbb{U}$, the only models which a is Lascar-independent of, are ones that realize only one of the formulas $v \sim a$ and $v \not\sim a$, and do not realize $v = a$ (i.e. do not contain a). In other words, $a \mathop{\smile\!\!\!\!\!\mid}^{sL} \mathcal{M}$ (or equivalently $a \mathop{\smile\!\!\!\!\!\mid}^{wL} \mathcal{M}$) if and only if $a \notin \mathcal{M}$ and $\forall b \in \mathcal{M} : \ a \sim b$ or $\forall b \in \mathcal{M} : \ b \sim a$.
Generalizing to a tuple $\overline{a} \subseteq \mathbb{U}$ of arbitrary (small) length, where w.l.o.g. the elements of $\overline{a}$ are distinct, $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{sL} \mathcal{M}$ ($\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{wL} \mathcal{M}$) if and only if for each element a_λ of $\overline{a}$, $a_\lambda \notin \mathcal{M}$, and either $\forall b \in \mathcal{M} : \ a_\lambda \sim b$ or $\forall b \in \mathcal{M} : \ a_\lambda \not\sim b$.
Finally, if $\overline{a} \in \mathbb{U}$ is a small tuple, and $X \subseteq \mathbb{U}$ a finite set, then $\overline{a} \mathop{\smile\!\!\!\!\!\mid}^{sL}_{X} \mathcal{M}$ if and only if none of the elements of $\overline{a} \setminus X$ are contained in $\mathcal{M}$, and for any $a_\lambda \in \overline{a} \setminus X$, the formula $v \sim a_\lambda$ is constant on $\mathcal{M} \setminus X$.

8.1 Heirs and Coheirs

A closely related notion is that of (co-)heirs of a type, see [17]. An heir of an A-type p is an extension $q \in S(B)$ to some set $B \supseteq A$, such that no more A-formulas are represented in q than in p. A coheir of p is an extension q which is 'finitely satisfiable' in A (this can only reasonably be expected to hold when A is a model).

Definition 8.11. For sets $A \subseteq B \subseteq \mathbb{U}$, and $p \in S(A)$, $q \in S(B)$ with $p \subseteq q$, we say that q is an *heir* of p if $C_A(q) \subseteq C_A(p)$.
We say that q is a *coheir* of p if for each $\varphi(\overline{v}) \in q$, there exists $\overline{a} \subseteq A$ s.t. $\mathbb{U} \models \varphi(\overline{a})$.

Remark 8.12. *It is not difficult to see that for $\overline{a}, \overline{b}, A \subset \mathbb{U}$,*

$$\mathrm{tp}(\overline{a}/A\overline{b}) \text{ is an heir of } \mathrm{tp}(\overline{a}/A) \iff \mathrm{tp}(\overline{b}/A\overline{a}) \text{ is a coheir of } \mathrm{tp}(\overline{b}/A).$$

The following guarantees existence of (co-)heirs of types over models. This is Proposition 69 in [16], which is left without proof there. The proof we give for the first part follows Lascar and Poizat [17, Proposition 3.2].

Proposition 8.13. *Let $B \supseteq A \supseteq \mathcal{M} \preccurlyeq \mathbb{U}$ and $q \in S(\mathbb{U}/A)$ an heir (resp. coheir) of $p \in S(\mathbb{U}/\mathcal{M})$. Then there exists an heir (resp. coheir) $r \in S(\mathbb{U}/B)$ of p with $q \subseteq r$.*

Proof. Suppose first that $q \in S_\kappa(A)$ is an heir of $p \in S_\kappa(\mathcal{M})$, and by saturation find a tuple $\overline{a} \in \mathbb{U}^\kappa$ realizing q. Expand to a language $\mathcal{L}'$ with new constant symbols $(c_b)_{b\in B}$ and a new tuple $\overline{d}$ of length κ. Define the set of $\mathcal{L}'_{\mathcal{M}}$-sentences

$$\Phi := \left\{\neg\varphi(\overline{d}, c_{\overline{b}}) \;\middle|\; \varphi \in \mathcal{L}_{\mathcal{M}},\ \varphi \notin C_{\mathcal{M}}(p),\ \overline{b} \subseteq B\right\}$$

Then $\Sigma := \Phi \cup$ "$\mathrm{tp}(\overline{d}/c_A) = q$" $\cup$ "$\mathrm{tp}((c_b)_{b\in B}/\mathcal{M}) = \mathrm{tp}(\overline{B}/\mathcal{M})$" is finitely satisfiable in $\mathbb{U}$. To see this, let $\neg\varphi(\overline{d}, c_{\overline{b}_1}) \in \Phi$, $\psi(\overline{v};\overline{x}) \in q$ (where $\overline{x} \subseteq A$) and $\overline{b}_2 \subseteq B$, $\theta \in \mathcal{L}_{\mathcal{M}}$ with $\theta(\overline{b}_2)$. We can assume w.l.o.g. that $\overline{b}_2 = \overline{x}\overline{b}_2'$ extends $\overline{x}$, where $\overline{b}_2' \subseteq B \setminus A$. Then,

$$\psi(\overline{v};\overline{x}) \ \wedge\ \exists\overline{w}\ \theta(\overline{xw}) \in q.$$

Since q is an heir of p by assumption, we can find $\overline{m} \subseteq \mathcal{M}$ with

$$\mathbb{U} \models \psi(\overline{a};\overline{m}) \ \wedge\ \exists\overline{w}\ \theta(\overline{mw}).$$

Now by $\mathcal{M} \preccurlyeq \mathbb{U}$ we can find $\overline{m}' \subseteq \mathcal{M}$ with $\theta(\overline{mm}')$. Finally, if $\overline{b}_1$ overlaps with $\overline{b}_2$, pick $\overline{m}_1 \subseteq \mathcal{M}$ accordingly (otherwise arbitrarily). Since $\varphi \notin C_{\mathcal{M}}(p)$, we get $\neg\varphi(\overline{a}, \overline{m}_1)$.
By compactness and saturation, we conclude that we can find $\overline{a}', A' \subseteq B'$ in $\mathbb{U}$ such that Σ is satisfied. Then $\mathrm{tp}(\overline{a}'/B')$ is an heir of p by Φ, $\mathrm{tp}(\overline{a}'/A') = q$, and $\mathrm{tp}(\overline{B}'/\mathcal{M}) = \mathrm{tp}(\overline{B}/\mathcal{M})$. We can find by homogeneity $f \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ with $f(\overline{B}') = \overline{B}$, and in particular $f(A') = A$. Then $\mathrm{tp}(f(\overline{a}')/B) = \mathrm{tp}(\overline{a}'/B')$ is an heir of p and $\mathrm{tp}(f(\overline{a}')/A) = \mathrm{tp}(\overline{a}'/A') = \mathrm{tp}(\overline{a}/A) = q$.
The argument for coheirs is similar. We add to $\mathcal{L}$ a κ-tuple of constants $\overline{d}$, let

$$\Phi := \left\{\neg\varphi(\overline{c}) \mid \overline{c} \subseteq \overline{d}, \varphi \in \mathcal{L}_B \text{ and } \nexists\overline{m} \subseteq \mathcal{M} :\ \varphi(\overline{m})\right\}$$

and observe that $\Phi \cup q(\overline{d})$ is finitely satisfiable in $\mathbb{U}$, since q is finitely satisfiable in $\mathcal{M}$. $\square$

Note that since any type is trivially a (co-)heir of itself, it follows from the above proposition that any $\mathcal{M}$-type has (co-)heirs in $S(A)$ for any $A \supseteq \mathcal{M}$.
Moreover, this guarantees existence of $\mathop{\smile\!\!\!\!\!\!|}^{wL}$and $\mathop{\smile\!\!\!\!\!\!|}^{sL}$over models, by the following connection between (co)heirs and Lascar-independence.

Proposition 8.14. *For $\mathcal{M} \preccurlyeq \mathbb{U}$ a model, $\mathop{\smile\!\!\!\!\!\!|}^{wL}_{\mathcal{M}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{sL}_{\mathcal{M}}$ coincide, and $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{wL}_{\mathcal{M}} \overline{b}$ iff $\operatorname{tp}(\overline{a}/\mathcal{M}\overline{b})$ is an heir of $\operatorname{tp}(\overline{a}/\mathcal{M})$, or equivalently, $\operatorname{tp}(\overline{b}/\mathcal{M}\overline{a})$ is a coheir of $\operatorname{tp}(\overline{b}/\mathcal{M})$.*

Proof. Clearly, the unique bound for $\operatorname{tp}(\overline{a}/\mathcal{M})$ is $C_{\mathcal{M}}(\overline{a}/\mathcal{M})$. Thus, any quasi-bound for $\operatorname{tp}(\overline{a}/\mathcal{M})$ is a bound, and therefore $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{wL}_{\mathcal{M}} \overline{b} \iff \overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL}_{\mathcal{M}} \overline{b}$.
It follows also that if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL}_{\mathcal{M}} \overline{b}$, then $\operatorname{tp}(\overline{a}/\mathcal{M}\overline{b})$ must be an heir of $\operatorname{tp}(\overline{a}/\mathcal{M})$.
On the other hand, if $\operatorname{tp}(\overline{a}/\overline{b}\mathcal{M})$ is an heir of $\operatorname{tp}(\overline{a}/\mathcal{M})$, and we fix some model $\mathcal{M}' \supseteq \overline{b}\mathcal{M}$, then by Proposition 8.13 and saturation, we can find $\overline{a}'$ s.t. $\operatorname{tp}(\overline{a}'/\mathcal{M}') \supseteq \operatorname{tp}(\overline{a}/\mathcal{M}\overline{b})$ is an heir of $\operatorname{tp}(\overline{a}/\mathcal{M})$. By homogeneity, there exists $f \in \operatorname{Aut}_{\mathcal{M}b}(\mathbb{U})$ with $f(\overline{a}') = \overline{a}$. Then $C_{\mathcal{M}}(\overline{a}/f(\mathcal{M}'))$ is a bound for $\operatorname{tp}(\overline{a}/\mathcal{M})$, and witnesses $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL}_{\mathcal{M}} \overline{b}$. □

This characterization of $\mathop{\smile\!\!\!\!\!\!|}^{sL}$ and $\mathop{\smile\!\!\!\!\!\!|}^{wL}$ in terms of (co)heirs, together with Proposition 8.16, yields the following corollary.

Corollary 8.15. *Let $\mathcal{M} \preccurlyeq \mathbb{U}$, and suppose $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{wL}_{\mathcal{M}} \overline{b}$. Then*

(i) for any $\overline{b}' \subset \mathbb{U}$ there exists $\overline{a}'$ such that $\operatorname{tp}(\overline{a}/\mathcal{M}\overline{b}) = \operatorname{tp}(\overline{a}'/\mathcal{M}\overline{b})$ and $\overline{a}' \mathop{\smile\!\!\!\!\!\!|}^{sL}_{\mathcal{M}} \overline{b}\overline{b}'$,

(ii) for any $\overline{a}' \subset \mathbb{U}$ there exists $\overline{b}'$ with $\operatorname{tp}(\overline{b}'/\mathcal{M}\overline{a}) = \operatorname{tp}(\overline{b}/\mathcal{M}\overline{a})$ and $\overline{a}\overline{a}' \mathop{\smile\!\!\!\!\!\!|}^{sL}_{\mathcal{M}} \overline{b}'$.

We prove now that $\mathop{\smile\!\!\!\!\!\!|}^{sL}$ satisfies a slightly stronger version existence, over any set X. Namely, for any model $\mathcal{M} \supseteq X$ we can find $\mathcal{M}'$ with $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL}_{X} \mathcal{M}'$ and $\operatorname{tp}(\overline{a}/\mathcal{M}') \ll_X \operatorname{tp}(\overline{a}/\mathcal{M})$. This is Proposition 64 in [16], in which an idea for a proof, using ultrafilters, is given. We prove this using Zorn's Lemma.

Proposition 8.16. *Let $\overline{a}, X \subseteq \mathbb{U}$, and $X \subseteq \mathcal{M} \preccurlyeq \mathbb{U}$. Then there exists $X \subseteq \mathcal{M}' \preccurlyeq \mathbb{U}$ such that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL}_{X} \mathcal{M}'$ and $\operatorname{tp}(\overline{a}/\mathcal{M}') \ll_X \operatorname{tp}(\overline{a}/\mathcal{M})$.*

Proof. We assume for ease of notation that $X = \emptyset$. The proof is completely analogous in the general case.
Let

$$P := \Big\{ \operatorname{tp}(\overline{a}/\mathcal{N}) \mid \mathcal{N} \preccurlyeq \mathbb{U} \text{ and } \operatorname{tp}(\overline{a}/\mathcal{N}) \ll \operatorname{tp}(\overline{a}/\mathcal{M}) \Big\}.$$

P is partially ordered by the fundamental order $\ll$. We will show via Zorn's Lemma that P has a minimal element w.r.t. $\ll$. This will prove the Proposition: If $q = \operatorname{tp}(\overline{a}/\mathcal{N})$ is a minimal element of P, then clearly $\overline{a} \mathop{\smile\!\!\!\!\!\!|}^{sL} \mathcal{N}$, and $q \ll \operatorname{tp}(\overline{a}/\mathcal{M})$.

Claim: For any $\ll$-chain $P_0 \subseteq P$, there is a lower bound for P_0 in P.
Let $P_0 \subseteq P$ be a chain, and define

$$\Phi := \left\{ \neg\exists\overline{v}\ \varphi(\overline{a},\overline{v}) \;\middle|\; \varphi(\overline{u},\overline{v}) \notin C(p) \text{ for some } p \in P_0 \right\}.$$

Then any finite subset of Φ is satisfied by some $\mathcal{N} \preccurlyeq \mathbb{U}$: Suppose $\varphi_1(\overline{u}_1,\overline{v}),\ldots,\varphi_n(\overline{u}_n,\overline{v}) \in \Phi$ and assume w.l.o.g. that $\overline{u}_1 = \cdots = \overline{u}_n$. Find for each $i = 1,\ldots,n$ some $p_i = \mathrm{tp}(\overline{a}/\mathcal{N}_i) \in P_0$ with $\varphi_i \notin C(p_i)$. Since P_0 is linearly ordered, we can find a $\ll$-smallest element p_k of $\{p_1,\ldots,p_n\}$. Then $\mathcal{N}_k \models \{\neg\exists\overline{v}\ \varphi_i(\overline{a},\overline{v}) \mid 1 \leq i \leq n\}$, so $\mathrm{tp}(\overline{a}/\mathcal{N}_k)$ is a lower bound for P_0 in P.
It follows by Zorn's Lemma that P has a $\ll$-minimal element. $\square$

8.2 Toward Lascar's Reconstruction Theorem

The motivation for introducing the notion of Lascar-independence, and the reason why it plays a central role in Lascar's Reconstruction Theorem, are two theorems which we shall prove in this section. Firstly, Theorem 8.22 states that if two small models $\mathcal{M}$ and $\mathcal{N}$ are weakly-Lascar-independent, then the group of Lascar-strong automorphisms is generated by $\mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$. In the case where T is $\aleph_0$-categorical and G-compact, we can reformulate this, in light of Corollary 7.14, as follows: Any finite equivalence relation that is definable over $\mathcal{M}$ and over $\mathcal{N}$ is definable. All results in this section other than Theorem 8.23 are in preparation for Theorem 8.22.
Secondly, Theorem 8.23 will provide a way to move from model-continuity of a functor $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ (i.e. preservation of unions of models in the sense of Definition 9.1) to continuity of $F : \mathrm{Aut}(\mathcal{M}_T) \to \mathrm{Aut}(F(\mathcal{M}_T))$ in the sense of pointwise convergence.

Lemma 8.17 ([16, Lemma 612]). *Suppose* $\mathrm{tp}(\overline{a}/\overline{b}) = \mathrm{tp}(\overline{a}'/\overline{b})$. *Then for any* $f' \in \mathrm{Aut}_{a'}(\mathbb{U})$ *there exist* $f \in \mathrm{Aut}_a(\mathbb{U})$ *and* $h \in \mathrm{Aut}_b(\mathbb{U})$ *such that* $f' = h^{-1} \circ f \circ h$.

Proof. By homogeneity, find $h \in \mathrm{Aut}_b(\mathbb{U})$ with $h(\overline{a}') = \overline{a}$. Then $hf'h^{-1} \in \mathrm{Aut}_a(\mathbb{U})$. $\square$

Lemma 8.18 ([16, Lemma 610]). *If* $\mathrm{tp}(\overline{a}/\mathcal{M}) = \mathrm{tp}(\overline{a}'/\mathcal{M})$ *and* $\overline{a}\overline{a}' \mathop{\smile\!\!\!\!\!\!\mid}^{wL}_{\mathcal{M}} \overline{b}$*, then* $\mathrm{tp}(\overline{a}/\mathcal{M}\overline{b}) = \mathrm{tp}(\overline{a}'/\mathcal{M}\overline{b})$.

Proof. Assume, for the sake of contradiction, there exists $\varphi(\overline{v},\overline{b},\overline{m}) \in \mathrm{tp}(\overline{a}/\mathcal{M}\overline{b}) \setminus \mathrm{tp}(\overline{a}'/\mathcal{M}\overline{b})$ (where $\overline{m} \subseteq \mathcal{M}$). Then $\varphi(\overline{a},\overline{v},\overline{m}) \wedge \neg\varphi(\overline{a}',\overline{v},\overline{m})$ is realized in $\mathcal{M}\overline{b}$, so by

$\overline{aa}' \mathop{\smile\!\!\!\!\!\!\,|}^{wL}_{\mathcal{M}} \overline{b}$ and Proposition 8.14, it must be realized by some $\overline{m}' \subseteq \mathcal{M}$ as well. Therefore, $\mathbb{U} \models \varphi(\overline{a}, \overline{m}', \overline{m}) \wedge \neg\varphi(\overline{a}', \overline{m}', \overline{m})$, but this is a contradiction to our assumption that $\mathrm{tp}(\overline{a}/\mathcal{M}) = \mathrm{tp}(\overline{a}'/\mathcal{M})$. $\square$

Denote by X some finite set in $\mathbb{U}$, and by $\mathcal{M}, \mathcal{M}', \mathcal{N}$ small models, containing X.

Lemma 8.19 ([16, Lemma 611])**.** *If* $\overline{a} \subset \mathbb{U}$ *with* $\mathrm{tp}(\overline{a}/\mathcal{M}) \ll_X \mathrm{tp}(\overline{a}/\mathcal{M}')$*, then there exists* $X \subseteq \mathcal{M}'' \preccurlyeq \mathbb{U}$ *with* $\mathrm{tp}(\mathcal{M}''/\overline{a}) = \mathrm{tp}(\mathcal{M}/\overline{a})$ *such that* $\mathrm{tp}(\overline{a}/\mathcal{M}'\mathcal{M}'')$ *is an heir of* $\mathrm{tp}(\overline{a}/\mathcal{M}')$*.*

Proof. Assume w.l.o.g. $X = \emptyset$ and let $\kappa := |\mathcal{M}|$.
Let$(c_\lambda)_{\lambda<\kappa}$ be new constants, and define a set of formulas

$$\Phi := \Big\{\neg\varphi(\overline{a}, \overline{x},\ c_{\overline{\lambda}}) \mid \overline{x} \in \mathcal{M}'^{<\omega},\ \overline{\lambda} \subseteq \kappa,\ \varphi \in \mathcal{L}_X \text{ and } \varphi(\overline{u}, \overline{v}, \overline{w}) \notin C_X(\overline{a}/\mathcal{M}')\Big\}.$$

Then $\Phi \cup$ "$\mathrm{tp}((c_\lambda)_{\lambda<\kappa}/\overline{a}) = \mathrm{tp}(\mathcal{M}/\overline{a})$" is finitely satisfiable, since for any $\psi(\overline{a}, \overline{v}) \in \mathrm{tp}(\mathcal{M}/\overline{a})$, we have $\psi(\overline{u}, \overline{v}) \in C_X(\overline{a}/\mathcal{M}) \subseteq C_X(\overline{a}/\mathcal{M}')$, so there exists $\overline{y} \subseteq \mathcal{M}'$ with $\psi(\overline{a}, \overline{y})$, and by definition, for any $\neg\varphi(\overline{a}, \overline{x}, c_{\overline{\lambda}}) \in \Phi$, we have $\neg\varphi(\overline{a}, \overline{x}, \overline{y})$. We conclude that there exists a set $C \subseteq \mathbb{U}$ such that $\mathrm{tp}(C/\overline{a}) = \mathrm{tp}(\mathcal{M}/\overline{a})$ and Φ is realized by C. In particular, C must be a model of T, and we write $\mathcal{M}'' := C$. By construction of Φ, we get $C_{\mathcal{M}'}(\overline{a}/\mathcal{M}'\mathcal{M}'') \subseteq C_{\mathcal{M}'}(\overline{a}/\mathcal{M}')$. $\square$

We prove now that whenever $\mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M}')$ is an heir of $\mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M})$, any $f \in \mathrm{Aut}_{\mathcal{M}'}(\mathbb{U})$ can be written as the composition of three elements of $\mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$.

Proposition 8.20 ([16, Proposition 613])**.** *Suppose* $\mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M}) \ll_X \mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M}')$*. Then for all* $f' \in \mathrm{Aut}_{\mathcal{M}'}(\mathbb{U})$*, there exists* $g_1, g_2 \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ *and* $f \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ *such that* $f' = g_1 f g_2$*.*

Proof. Let $\mathcal{M}, \mathcal{M}', \mathcal{N}$ and f' be given as in the proposition statement.
By Lemma 8.19, there exists $X \subseteq \mathcal{M}'' \preccurlyeq \mathbb{U}$ with $\mathrm{tp}(\overline{\mathcal{M}}''/\mathcal{N}) = \mathrm{tp}(\overline{\mathcal{M}}/\mathcal{N})$ such that $\mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M}'\mathcal{M}'')$ is an heir of $\mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M}')$, or equivalently, $\overline{\mathcal{N}} \mathop{\smile\!\!\!\!\!\!\,|}^{sL}_{\mathcal{M}'} \mathcal{M}''$. Let $\mathcal{N}' := f'(\mathcal{N})$. Then $\mathrm{tp}(\overline{\mathcal{N}}'/\mathcal{M}') = \mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M}')$, and by point (ii) of Corollary 8.15 there exists a model $\mathcal{M}_1$ with $\mathrm{tp}(\mathcal{M}_1/\mathcal{N}\mathcal{M}') = \mathrm{tp}(\mathcal{M}''/\mathcal{N}\mathcal{M}')$ and

$$\overline{\mathcal{N}\mathcal{N}}' \mathop{\smile\!\!\!\!\!\!\,|}^{sL}_{\mathcal{M}'} \mathcal{M}_1.$$

Thus, we can apply Lemma 8.18 to get

$$\mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M}'\mathcal{M}_1) = \mathrm{tp}(\overline{\mathcal{N}}'/\mathcal{M}'\mathcal{M}_1).$$

By homogeneity, it follows that we can find $\alpha \in \mathrm{Aut}_{\mathcal{M}_1}(\mathbb{U})$ with $\alpha(\overline{\mathcal{N}}) = \overline{\mathcal{N}}'$ pointwise. Now, since

$$\mathrm{tp}(\overline{\mathcal{M}}_1/\mathcal{N}) = \mathrm{tp}(\overline{\mathcal{M}}''/\mathcal{N}) = \mathrm{tp}(\overline{\mathcal{M}}/\mathcal{N})$$

we may find by Lemma 8.17, $f \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ and $h \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ with $\alpha = h^{-1}fh$. Setting $g_1 := h^{-1}$ and $g_2 := h\alpha^{-1}f'$, we get $g_1 \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$, and

$$g_2(\overline{\mathcal{N}}) = (h \circ \alpha^{-1})(\overline{\mathcal{N}}') = h(\overline{\mathcal{N}}) = \overline{\mathcal{N}}$$

so $g_2 \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ as well. Finally, we see that

$$g_1 \circ f \circ g_2 = (h^{-1}) \circ (h\alpha h^{-1}) \circ (h\alpha^{-1}f') = f'$$

which concludes the proof. □

Next we show that if $\mathcal{M}$ and $\mathcal{N}$ are **strongly**-Lascar-independent, then not just can any $f \in \mathrm{Aut}_{\mathcal{M}'}(\mathbb{U})$ be written as a composition of elements of $\mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$, but in fact, as a composition of five such automorphisms.

Theorem 8.21 ([16, Theorem 614]). *Suppose* $\overline{\mathcal{N}} \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{X} \mathcal{M}$. *Then for any* $f' \in \mathrm{Aut}_{\mathcal{M}'}(\mathbb{U})$, *there exist* $g_1, g_2, g_3 \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ *and* $h_1, h_2 \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ *such that* $f' = g_1h_1g_2h_2g_3$.

Proof. Clearly $\overline{\mathcal{N}} \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{\mathcal{M}} \mathcal{M}$, so by point (i) of Corollary 8.15 there exists $\mathcal{N}'$ with $\mathrm{tp}(\mathcal{N}'/\mathcal{M}) = \mathrm{tp}(\mathcal{N}/\mathcal{M})$ and $\mathcal{N}' \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{\mathcal{M}} \mathcal{M}'\mathcal{M}$, i.e. $\mathcal{N}' \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{\mathcal{M}} \mathcal{M}'$ by normality of $\mathop{\smile\!\!\!\!\!\!\mid}^{sL}$. Thus, we have $\mathrm{tp}(\mathcal{N}'/\mathcal{M}') \ll_X \mathrm{tp}(\mathcal{N}'/\mathcal{M}) = \mathrm{tp}(\mathcal{N}/\mathcal{M})$, so since $\mathcal{N} \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{X} \mathcal{M}$ by assumption, we get $\mathrm{tp}(\mathcal{N}'/\mathcal{M}) \ll_X \mathrm{tp}(\mathcal{N}'/\mathcal{M}')$.
It follows by Proposition 8.20 that there exist $\epsilon_1, \epsilon_2 \in \mathrm{Aut}_{\mathcal{N}'}(\mathbb{U})$ and $f \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ with

$$f' = \epsilon_1 \circ f \circ \epsilon_2. \tag{8.3}$$

Since $\mathrm{tp}(\overline{\mathcal{N}}'/\mathcal{M}) = \mathrm{tp}(\overline{\mathcal{N}}/\mathcal{M})$ we can find $\delta \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ with $\delta(\overline{\mathcal{N}}') = \overline{\mathcal{N}}$. Setting

$$g_1 := \delta^{-1},\ h_1 := \delta \circ \epsilon_1 \circ \delta^{-1},\ g_2 := \delta \circ f \circ \delta^{-1},\ h_2 := \delta \circ \epsilon_2 \circ \delta^{-1},\ \text{and } g_3 := \delta,$$

it is not difficult to see that $g_i \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$ and $h_i \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ for all i, and that

$$g_1 \circ h_1 \circ g_2 \circ h_2 \circ g_3 = \epsilon_1 \circ f \circ \epsilon_2 \overset{(8.3)}{=} f'.$$

□

Finally, we use the connection between strong- and weak-Lascar-independence to prove the first of two central result of this chapter.

Theorem 8.22 ([16, Theorem 615]). *Suppose $\overline{\mathcal{N}} \mathop{\smile\!\!\!\!\!\!\,|}^{wL}_{X} \mathcal{M}$. Then*

$$\mathrm{Autf_L}(\mathbb{U}/X) = \big\langle\, \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \big\rangle.$$

Proof. We assume w.l.o.g. that $X = \emptyset$.
Let $\mathcal{M}' \preccurlyeq \mathbb{U}$ and $f' \in \mathrm{Aut}_{\mathcal{M}'}(\mathbb{U})$. We will show that $f' \in \big\langle\, \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \big\rangle$.
Since $\mathcal{N} \mathop{\smile\!\!\!\!\!\!\,|}^{wL} \mathcal{M}$, we may find by (the proof of) Lemma 8.5 for each finite $\Phi \subseteq C(\overline{\mathcal{N}}/\mathcal{M})$ a model $X \subseteq \mathcal{M}_\Phi \preccurlyeq \mathbb{U}$ s.t. $\mathcal{N} \mathop{\smile\!\!\!\!\!\!\,|}^{sL} \mathcal{M}_\Phi$ and $F \subseteq C(\overline{\mathcal{N}}/\mathcal{M}_\Phi)$. Moreover, we may assume, by downward Löwenheim-Skolem that $|\mathcal{M}_\Phi| = |\mathcal{M}|$.
Expand the language $\mathcal{L}$ by unary function symbols $F, G_1, G_2, G_3, H_1, H_2$ and a unary relation symbol P, and denote the expanded language by $\mathcal{L}'$. Moreover, let $(c_x)_{x\in\mathbb{U}}$ be new constant symbols. Consider the union Σ of the following sets of $\mathcal{L}'_{\mathbb{U}}$-sentences:

(i) The set Σ_0 stating that $\mathbb{U}$ is an $\mathcal{L}$-substructure and Σ_1 stating that P defines a submodel, i.e.

$$\begin{aligned}\Sigma_0 :=& \{\alpha(c_{\overline{x}}) = c_{\alpha^{\mathbb{U}}(\overline{x})} \mid \alpha \text{ fct. in } \mathcal{L},\ \overline{x} \subset \mathbb{U}\} \cup \\ & \{R(c_{\overline{x}}) \mid R \text{ rel. in } \mathcal{L},\ \overline{x} \in R(\mathbb{U})\} \cup \{d = c_{d^{\mathbb{U}}} \mid d \text{ const. in } \mathcal{L}\} \\ \Sigma_1 :=& \big\{\forall\overline{v}\exists\overline{w}\ P(\overline{v},\overline{w}) \wedge \varphi(\overline{v},\overline{w}) \mid T \vdash \forall\overline{v}\exists\overline{w}\ \varphi(\overline{v},\overline{w})\big\}.\end{aligned}$$

(ii) The set Σ_3 stating that $F, G_1, G_2, G_3, H_1, H_2$ are $\mathcal{L}$-automorphisms, and that $F = G_1H_1G_2H_2G_3$, i.e.

$$\begin{aligned}\Sigma_2 =& \big\{\varphi(\overline{v}) \leftrightarrow \varphi(F(\overline{v})) \leftrightarrow \varphi(G_1(\overline{v})) \leftrightarrow \cdots \leftrightarrow \varphi(H_2(\overline{v})) \;\big|\; \varphi \in \mathcal{L}\big\} \cup \\ & \big\{\forall v \exists \overline{w}\ (v = F(w_1) = G_1(w_2) = \cdots = H_2(w_6))\big\} \cup \\ & \big\{\forall v :\ F(v) = (G_1H_1G_2H_2G_3)(v)\big\}.\end{aligned}$$

(iii) The set Σ_4 stating that G_1, G_2, G_3 fix the elements of P pointwise, and that H_1, H_2 fix $(c_n)_{n\in\mathcal{N}}$ pointwise, i.e.

$$\Sigma_3 := \big\{\forall v\ (P(v) \to G_i(v) = v) \;\big|\; i = 1,2,3\big\} \cup \big\{H_i(c_n) = c_n \;\big|\; i = 1,2,\ n \in \mathcal{N}\big\}.$$

(v) The set Σ_4 stating that $F|_{\mathbb{U}} = f'$, i.e.

$$\Sigma_4 := \big\{F(d_x) = d_{f'(x)} \;\big|\; x \in \mathbb{U}\big\}.$$

(v) The set Σ_5 stating that $C(\overline{\mathcal{N}}/\mathcal{M}) \subseteq C(\overline{\mathcal{N}}/P)$, i.e.

$$\Sigma_5 := \big\{\, \underbrace{\exists\overline{v}\ (P(\overline{v}) \wedge \varphi(c_{\overline{x}},\overline{v}))}_{=:\psi_\varphi} \;\big|\; \varphi \in \mathcal{L}, \overline{x} \subset \mathcal{N} \text{ and } \mathcal{M} \models \exists\overline{v}\ \varphi(\overline{x},\overline{v})\big\}.$$

We show that $\Sigma \cup T$ is finitely satisfiable. For a finite $\Phi \subseteq C(\overline{\mathcal{N}}/\mathcal{M})$, we have $\Phi \subseteq C(\overline{\mathcal{N}}/\mathcal{M}_\Phi)$ and since $\overline{\mathcal{N}} \mathop{\smile\!\!\!\!\!\!\mid}^{sL} \mathcal{M}_\Phi$, there exist by Theorem 8.21 $g_1^\Phi, g_2^\Phi, g_3^\Phi \in \mathrm{Aut}_{\mathcal{M}_\Phi}(\mathbb{U})$ and $h_1^\Phi, h_2^\Phi \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})$ with $f' = g_1^\Phi h_1^\Phi g_2^\Phi h_2^\Phi g_3^\Phi$. Thus,

$$\left\langle \mathbb{U}, f', g_1^\Phi, g_2^\Phi, g_3^\Phi, h_1^\Phi, h_2^\Phi, \mathcal{M}_\Phi \right\rangle \models \Sigma_1 \cup \cdots \cup \Sigma_5 \cup \{\psi_\varphi \mid \varphi \in \Phi\}.$$

By Compactness, we may find

$$\left\langle \mathbb{U}^+, f^+, g_1^+, g_1^+, g_3^+, h_1^+, h_1^+, \mathcal{M}^+ \right\rangle \models \Sigma \cup T.$$

In fact, we may assume that $\mathbb{U}^+$ is saturated as a model of T. Otherwise, we could find a larger saturated model and extend the automorphisms by homogeneity.
Now, $\mathbb{U} \overset{\mathcal{L}}{\preccurlyeq} \mathbb{U}^+$ and $f^+ = g_1^+ h_1^+ g_2^+ h_2^+ g_3^+ \in \langle \mathrm{Aut}_{\mathcal{M}^+}(\mathbb{U}^+) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}^+) \rangle$ with $f^+|_{\mathbb{U}} = f'$. Moreover, $C(\overline{\mathcal{N}}/\mathcal{M}) \subseteq C(\overline{\mathcal{N}}/\mathcal{M}^+)$ by construction. By Proposition 8.20, it follows that in fact,

$$f^+ \in \langle \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}^+) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}^+) \rangle.$$

Thus, w.l.o.g. f^+ has the form $f^+ = g^+ \circ h^+$ for some $g^+ \in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}^+)$ and $h^+ \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}^+)$. By Lemma 2.14, there exists a saturated model $\mathbb{U}_1 \preccurlyeq \mathbb{U}^+$ of cardinality $|\mathbb{U}|$ with $\mathcal{M} \cup f'(\mathcal{M}) \cup \mathcal{N} \subseteq \mathbb{U}_1$, such that $g^+|_{\mathbb{U}_1}, h^+|_{\mathbb{U}_1} \in \mathrm{Aut}(\mathbb{U}_1)$. Since $\mathbb{U}$ and $\mathbb{U}_1$ are saturated models of the same cardinality containing $\mathcal{M}f'(\mathcal{M})\mathcal{N}$, we can find $\delta \in \mathrm{Aut}_{\mathcal{M}f'(\mathcal{M})\mathcal{N}}(\mathbb{U}^+)$ with $\delta(\mathbb{U}_1) = \mathbb{U}$ pointwise.
Now, we have $f := (\delta f^+ \delta^{-1})|_{\mathbb{U}} \in \mathrm{Aut}(\mathbb{U})$ and

$$\delta \circ f^+ \circ \delta^{-1} = \overbrace{(\delta g^+ \delta^{-1})}^{\in \mathrm{Aut}_{\mathcal{N}}(\mathbb{U})} \circ \overbrace{(\delta h^+ \delta^{-1})}^{\in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})}$$

so $f \in \langle \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}) \rangle$. Finally observe that since δ fixes $\mathcal{M}$ and $f'(\mathcal{M}) = f^+(\mathcal{M})$ pointwise, we have $f|_{\mathcal{M}} = f^+|_{\mathcal{M}} = f'|_{\mathcal{M}}$, and so $f' \in \langle \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{N}}(\mathbb{U}) \rangle$.
$\square$

We conclude this chapter with the second important result concerning Lascar independence that we will use in the proof of one of the final steps of Lascar's Independence Theorem. We want to build from an infinite set A an ascending sequence of models $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2 \preccurlyeq \ldots$, each of which is weakly-Lascar independent of A over larger and larger finite subsets of A. The following theorem makes this possible.

Theorem 8.23 ([16, Theorem 616]). *Let $\overline{A} \in \mathbb{U}^\omega$ be an infinite tuple, and write, for each $n \in \omega$, $\overline{A}_n := (a_1, \ldots, a_n)$. Then there exists a sequence $(M_n)_{n \in \omega}$ of submodels $\mathcal{M}_n \preccurlyeq \mathbb{U}$ of size $\kappa := \max(|\mathcal{L}|, \aleph_0)$ s.t. for any $n \in \omega$,*

1. $A_n \subset \mathcal{M}_n$,

2. $\mathcal{M}_n \preccurlyeq \mathcal{M}_{n+1}$,

3. $\overline{A} \mathop{\smile\!\!\!\!\!\!\mid}^{wL}_{A_n} \mathcal{M}_n$.

Proof. We proceed in three steps.
Step 1: For each $N \in \omega$ we construct countable models $\mathcal{M}_1^N, \ldots, \mathcal{M}_N^N \preccurlyeq \mathbb{U}$ such that

(i) $A_n \subset \mathcal{M}_n^N$,

(ii) $C_{A_{n-1}}(\overline{A}/\mathcal{M}_{n-1}^N) \subseteq C_{A_n}(\overline{A}/\mathcal{M}_n^N)$,

(iii) $\overline{A} \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{A_n} \mathcal{M}_n^N$.

Fix $N \in \omega$. We construct $\mathcal{M}_N^N, \mathcal{M}_{N-1}^N, \cdots, \mathcal{M}_0^N$ step by step, in that order.
Firstly, by *existence* of $\mathop{\smile\!\!\!\!\!\!\mid}^{wL}$ (Proposition 8.16) and downward Löwenheim-Skolem, we can find a model $\mathcal{M}_N^N \supseteq a_N$ of size κ with $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{A_N} \mathcal{M}_N^N$.
Now, suppose $0 < n < N$, and $\mathcal{M}_N^N, \cdots, \mathcal{M}_{n+1}^N$ have already been constructed, with the desired properties. Again, by Proposition 8.16 and downward L-S, find $\mathcal{M}_n^N \supseteq a_n$ of size κ with $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{A_n} \mathcal{M}_n^N$ and $C_{A_n}(\overline{a}/\mathcal{M}_n^N) \subseteq C_{A_n}(\overline{A}/\mathcal{M}_{n+1}^N)$. Note that clearly, $C_{A_n}(\overline{A}/\mathcal{M}_{n+1}^N) \subseteq C_{A_{n+1}}(\overline{a}/\mathcal{M}_{n+1}^N)$, so we are done.

Step 2: We now find models $(\mathcal{M}'_n)_{n\in\omega}$ s.t. for each $n \in \omega$, $\overline{A} \mathop{\smile\!\!\!\!\!\!\mid}^{wL}_{A_n} \mathcal{M}'_n$ and $C_{A_n}(\overline{A}/\mathcal{M}'_n) \subseteq C_{A_{n+1}}(\overline{A}/\mathcal{M}'_{n+1})$.
For $n \in \omega$, define sets Σ_n and Σ'_n of $\mathcal{L}_{A_n}$-formulas, as follows

$$\Sigma_n := \left\{\exists \overline{w}\ \varphi(\overline{A}, \overline{w}) \;\middle|\; \varphi \in \mathcal{L}_{A_n} \text{ and } \exists N_0 \geq n \forall N \geq N_0 :\ \varphi \in C_{A_n}(\overline{A}/\mathcal{M}_n^N)\right\}$$
$$\Sigma'_n := \left\{\nexists \overline{w}\ \varphi(\overline{A}, \overline{w}) \;\middle|\; \varphi \in \mathcal{L}_{A_n} \text{ and } \exists N_0 \geq n \forall N \geq N_0 :\ \varphi \notin C_{A_n}(\overline{A}/\mathcal{M}_n^N)\right\}.$$

Note that if $\mathcal{M}'_n \models \Sigma'_n$, then by construction,

$$\varphi \in C_{A_n}(\overline{A}/\mathcal{M}'_n) \implies \forall N_0 \geq n \exists N \geq N_0 :\ \varphi(\overline{v}, \overline{w}) \in C_{A_n}(\overline{A}/\mathcal{M}_n^N)$$

and since by Step 1, $\overline{A} \mathop{\smile\!\!\!\!\!\!\mid}^{sL}_{A_n} \mathcal{M}_n^N$ for each $N \geq n$, Lemma 8.5 tells us that $\overline{A} \mathop{\smile\!\!\!\!\!\!\mid}^{wL}_{A_n} \mathcal{M}'_n$.
Moreover, $\Phi_n \subseteq \Phi_{n+1}$ for all $n \in \omega$. We shall now construct iteratively the sequence $\mathcal{M}'_n$ s.t. $\mathcal{M}'_n \models \Sigma_n \cup \Sigma'_n$ and $C_{A_n}(\overline{A}/\mathcal{M}'_n) \subseteq C_{A_{n+1}}(\overline{A}/\mathcal{M}'_{n+1})$.
Firstly, note that $\Sigma_0 \cup \Sigma'_0 \cup T$ is finitely satisfiable. Both Σ_0 and Σ'_0 are closed under finite conjunctions. Thus, it is enough to observe that for any $\psi \in \Sigma_0$ and $\psi' \in \Sigma'_0$, we can find some N_0 such that for any $N \geq N_0$, $\mathcal{M}_0^N$ models ψ and ψ'. Thus, by Compactness and downward L-S, we may find a model $\mathcal{M}'_0 \models T \cup \Sigma_0 \cup \Sigma'_0$ of size κ.

Suppose now that $\mathcal{M}'_0, \dots, \mathcal{M}'_{n-1}$ have already been constructed. Consider the set Σ of $\mathcal{L}_A$-sentences

$$\Sigma := \left\{ \exists \overline{v}\ \varphi(\overline{A}, \overline{v}) \;\middle|\; \varphi(\overline{u}, \overline{v}) \in C_{A_{n-1}}(\overline{A}/\mathcal{M}'_{n-1}) \right\}.$$

Then $T \cup \Sigma_n \cup \Sigma'_n \cup \Sigma$ is finitely satisfiable: As before, we see that for any $\psi \in \Sigma_n$ and $\psi' \in \Sigma'_n$, there exists $N_0 \geq n$ such that ψ and ψ' are modelled by all $\mathcal{M}_n^N$ with $N \geq N_0$. Now, if $\varphi \in C_{A_{n-1}}(\overline{A}/\mathcal{M}'_{n-1})$, then since $\mathcal{M}'_{n-1} \models \Sigma_{n-1}$, we know that there exists $N^* \geq N_0$ such that $\varphi \in C_{A_{n-1}}(\overline{A}/\mathcal{M}_{n-1}^{N^*}) \subseteq C_{A_n}(\overline{A}/\mathcal{M}_n^{N^*})$.
Again by Compactness and downward L-S, we find $\mathcal{M}'_n \preccurlyeq \mathbb{U}$ of size κ that models $\Sigma_n \cup \Sigma'_n \cup \Sigma$, and by construction of Σ, $C_{A_{n-1}}(\overline{A}/\mathcal{M}'_n) \supseteq C_{A_{n-1}}(\overline{A}/\mathcal{M}'_{n-1})$.
Denote $\Phi_n := C_{A_n}(\overline{A}/\mathcal{M}'_n)$ for each $n \in \omega$.

Step 3: Finally, we will construct models $(\mathcal{M}_n)_{n\in\omega}$ of size κ such that $C_{a_n}(\overline{a}/\mathcal{M}_n) = \Phi_n$, $a_n \subset \mathcal{M}_n$ and $\mathcal{M}_n \preccurlyeq \mathcal{M}_{n+1}$ for each $n \in \omega$. Then by Step 2, we are done.
First, define $\mathcal{M}_0 := \mathcal{M}'_0$.
Suppose $\mathcal{M}_0, \dots, \mathcal{M}_n$ have already been constructed, with the desired properties. Then we have

$$C_{A_n}(\overline{A}/\mathcal{M}_n) = C_{A_n}(\overline{A}/\mathcal{M}'_n) \overset{\textit{Step 2}}{\subseteq} C_{A_n}(\overline{A}/\mathcal{M}'_{n+1})$$

so by Lemma 8.19, there exists $\mathcal{M}''_n \preccurlyeq \mathbb{U}$ such that $\mathrm{tp}(\mathcal{M}''_n/\overline{A}) = \mathrm{tp}(\mathcal{M}_n/\overline{A})$ and $\overline{A} \mathop{\smile\!\!\!\!\!\!|}^{sL}_{\mathcal{M}'_{n+1}} \mathcal{M}''_n$. By definition, we can find a model $\mathcal{M}''_{n+1} \supseteq \mathcal{M}''_n \cup \mathcal{M}'_{n+1}$ such that $C_{\mathcal{M}'_{n+1}}(\overline{A}/\mathcal{M}''_{n+1})$ is a bound of $\mathrm{tp}(\overline{A}/\mathcal{M}'_{n+1})$, so

$$C_{\mathcal{M}'_{n+1}}(\overline{A}/\mathcal{M}''_{n+1}) = C_{\mathcal{M}'_{n+1}}(\overline{a}/\mathcal{M}'_{n+1})$$

and in particular, since $A_{n+1} \subseteq \mathcal{M}'_{n+1}$,

$$C_{A_{n+1}}(\overline{A}/\mathcal{M}''_{n+1}) = C_{A_{n+1}}(\overline{A}/\mathcal{M}'_{n+1}) = \Phi_{n+1}.$$

Finally, since $\mathrm{tp}(\mathcal{M}_n/A) = \mathrm{tp}(\mathcal{M}''_n/A)$, we can find $\delta \in \mathrm{Aut}_A(\mathbb{U})$ with $\delta(\overline{\mathcal{M}''_n}) = \overline{\mathcal{M}_n}$. Set $\overline{\mathcal{M}_{n+1}} := \delta(\overline{\mathcal{M}''_{n+1}})$. Then we still have $A_{n+1} \subseteq \mathcal{M}_{n+1}$ and $\Phi_{n+1} = C_{A_{n+1}}(\overline{A}/\mathcal{M}_{n+1})$. Additionally, since $\mathcal{M}''_n \subseteq \mathcal{M}''_{n+1}$,

$$\mathcal{M}_n = \delta(\mathcal{M}''_n) \preccurlyeq \delta(\mathcal{M}''_{n+1}) = \mathcal{M}_{n+1}.$$

□

Chapter 9

Reconstructing T from Mod(T)

We have built the necessary tools to prove what we call the Lascar Reconstruction Theorem: Fix two theories T and T'. If T is $\aleph_0$-categorical and G-finite, then $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ implies bi-interpretability $T \simeq T'$. The strategy for the proof is as follows.
We want to ultimately show that for any given an equivalence $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ the restriction $F : \mathrm{Aut}(\mathcal{M}_T) \to \mathrm{Aut}(F(\mathcal{M}_T))$ is a continuous isomorphism. By a result from Section 4, T' will also be $\aleph_0$-categorical and $F(\mathcal{M}_T)$ is countable, so the MCAZ Theorem (5.11) will yield the bi-interpretation $T \simeq T'$.
Rather than investigating functors $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$, we will consider generally functors $\mathbf{Mod}(T) \to \mathbf{Set}$. We begin by defining in Section 9.1 the property of being *model-continuous* ('continuous' in [16]) for such a functor, which is a form of preservation of colimits. As we should expect, equivalences $\mathbf{Mod}(T) \to \mathbf{Mod}(T')$ have this property of model-continuity (Lemma 9.2). In Section 9.2, we consider functors $F : \mathbf{Mod}(T) \to \mathbf{Set}$, which have *finite supports* on countable models, that is, where the restriction $F : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Sym}(F(\mathcal{M}))$ is continuous w.r.t. pointwise convergence, for all models $\mathcal{M} \in \mathbf{Mod}(T)$. The key result of this chapter is Theorem 9.11 of Section 9.3, where we show that model-continuous functors $F : \mathbf{Mod}(T) \to \mathbf{Set}$ are *Lascar-continuous* (or *L-continuous*). This is where we make use of the work on Lascar-independence, in particular, Theorem 8.22. L-continuity of F states that for a model $\mathcal{M} \preccurlyeq \mathbb{U}$ and any point b in the the image of the embedding $F(\subseteq) : F(\mathcal{M}) \to F(\mathbb{U})$, there is a finite set $X \subseteq \mathcal{M}$ such that $F(f)(b) = b$ for all $f \in \mathrm{Autf_L}(\mathbb{U}/X)$. We apply Corollary 7.18 to see that if T is $\aleph_0$-categorical and G-finite, then L-continuous functors have the f.s.p. This will yield what is Corollary 714 in [16]: If $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ is an equivalence and T is G-finite and $\aleph_0$-categorical, then F has the f.s.p. - i.e. is continuous as a map $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(F(\mathcal{M}))$ for all models $\mathcal{M} \preccurlyeq \mathbb{U}$.

9.1 Model-Continuous Functors

We will denote throughout this section by F a functor from the category of models $\mathbf{Mod}(T)$ to the category of sets $\mathbf{Set}$. That is, F maps each model $\mathcal{M} \models T$ to a set $F(\mathcal{M})$, and every elementary map $f : \mathcal{M} \to \mathcal{M}'$ to a function $F(f) : F(\mathcal{M}) \to F(\mathcal{M}')$, preserving identities and composition.
We are interested specifically in functors that preserve what category theorists call *directed colimits*. The reader need not be familiar with this term, as we will define the corresponding property for the relevant case where F is a functor from $\mathbf{Mod}(T)$ to $\mathbf{Set}$, and we will refer to functors satisfying this property as *model-continuous*.

As before, we fix a saturated model $\mathbb{U} \models T$ of regular cardinality $> 2^{|T|}$ (in this case $|T| = \aleph_0$ by $\aleph_0$-categoricity). Moreover, for any $\mathcal{M} \preccurlyeq \mathbb{U}$, we shall denote by $\iota(\mathcal{M})$ for the inclusion map $\subseteq : \mathcal{M} \to \mathbb{U}$.

For an elementary chain of models of $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \dots$ of length λ, recall that we can canonicaly define a structure $\bigcup_{\kappa<\lambda} \mathcal{M}_\kappa$ that is itself a model of T (see Section 4).

Definition 9.1. We say that a functor $F : \mathbf{Mod}(T) \to \mathbf{Set}$ is *model-continuous* (w.r.t. $\mathbb{U}$) if for any countable elementary chain $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \dots \preccurlyeq \mathbb{U}$ of countable models, and $\mathcal{M}^* := \bigcup_n \mathcal{M}_n$, we have

$$\mathrm{im}\left[F(\iota_{\mathcal{M}^*})\right] = \bigcup_{n\in\omega} \mathrm{im}\left[F(\iota_{\mathcal{M}_n})\right].$$

Suppose T' is another $\aleph_0$-categorical theory. If we have an equivalence $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$, we may define a functor $\tilde{F} : \mathbf{Mod}(T) \to \mathbf{Set}$, which maps a model $\mathcal{M} \models T$ to $F(\mathcal{M})$, and then maps this structure to its universe, as an object in $\mathbf{Set}$. In categorical terms, $\tilde{F}$ is the result of composing F on the left with the *forgetful functor* $\mathbf{Mod}(T') \to \mathbf{Set}$. The name 'forgetful' stems from the fact that this functor simply takes a model, forgets all of its structure, and only keeps the underlying set. Elementary maps are mapped to themselves under the forgetful functor. That is, formally, for two models $\mathcal{M}, \mathcal{N} \models T$, the map $\tilde{F} : \mathrm{Hom}_{\mathbf{Mod}(T)}(\mathcal{M}, \mathcal{N}) \to F(\mathcal{N})^{F(\mathcal{M})}$ is the composition of $F : \mathrm{Hom}_{\mathbf{Mod}(T)}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}_{\mathbf{Mod}(T')}(F\mathcal{M}, F\mathcal{N})$ with the inclusion map $\mathrm{Hom}_{\mathbf{Mod}(T')}(F\mathcal{M}, F\mathcal{N}) \subseteq F(\mathcal{N})^{F(\mathcal{M})}$. This is sensible, since elementary maps between models are just maps between the universes, with additional properties.
With this definition in mind, we can extend the definition of a model-continuous functor, and we say $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ is *model-continuous* if $\tilde{F} : \mathbf{Mod}(T) \to \mathbf{Set}$ is model-continuous.

We prove now that equivalences $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ are model-continuous. This is a direct consequence of Proposition 4.8.

Lemma 9.2. *Any equivalence of categories $F : \boldsymbol{Mod}(T) \to \boldsymbol{Mod}(T')$ is model-continuous.*

Proof. Let $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \cdots \preccurlyeq \mathbb{U}$ be a countable elementary chain of countable models of T. Set $\mathcal{M}^* := \bigcup_n \mathcal{M}_n$, and let $\subseteq_n \colon \mathcal{M}_n \to \mathcal{M}^*$, and $\subseteq_{n,n+1} \colon \mathcal{M}_n \to \mathcal{M}_{n+1}$ be the respective inclusion maps. Then by Proposition 4.8, we get

$$F(\mathcal{M}^*) = \bigcup_{n\in\omega} \operatorname{im}\big[F(\subseteq_n)\big],$$

and therefore

$$\operatorname{im}\big[F(\iota_{\mathcal{M}^*})\big] = \bigcup_{n\in\omega} \operatorname{im}\big[F(\iota_{\mathcal{M}^*}\circ \subseteq_n)\big] = \bigcup_{n\in\omega} \operatorname{im}\big[F(\iota_n)\big]$$

where the last step holds since

$$\iota_{\mathcal{M}^*}\circ \subseteq_n \;=\; \subseteq(\mathcal{M}^*, \mathbb{U}) \circ \subseteq(\mathcal{M}_n, \mathcal{M}^*) \;=\; \subseteq(\mathcal{M}_n, \mathbb{U}) \;=\; \iota_n.$$

We conclude that F is model-continuous. $\square$

9.2 The Finite Support Property

Given a functor $F : \mathbf{Mod}(T) \to \mathbf{Set}$ and a model $\mathcal{M} \models T$, the map $F|_{\mathcal{M}} : \mathbf{End}(\mathcal{M}) \to F(\mathcal{M})^{F(\mathcal{M})}$ is a homomorphism from the monoid of elementary maps of $\mathcal{M}$ to the monoid of functions $F(\mathcal{M}) \to F(\mathcal{M})$. In particular, if we restrict F to $\operatorname{Aut}(\mathcal{M})$, then - because F is homomorphic, and thus maps bijections to bijections - we get a group-homomorphism $F|_{\operatorname{Aut}(\mathcal{M})} : \operatorname{Aut}(\mathcal{M}) \to \operatorname{Sym}(F(\mathcal{M}))$. Recall that both $\operatorname{Aut}(\mathcal{M})$ and $\operatorname{Sym}(\mathcal{M})$ are equipped with the topology of pointwise convergence, which is generated by cosets of pointwise stabilizers, i.e. sets of the form $\{f : f(\overline{a}) = \overline{b}\}$ where $\overline{a}$ and $\overline{b}$ are finite tuples. A homomorphism $\gamma : \operatorname{Aut}(\mathcal{M}) \to \operatorname{Sym}(F(\mathcal{M}))$ is continuous if and only if for any point $a \in F(\mathcal{M})$, there is a finite set of points $X \subseteq \mathcal{M}$ such that any $f \in \operatorname{Aut}_X(\mathcal{M})$ gets mapped to an element $\gamma(f) \in \operatorname{Sym}_a(F(\mathcal{M}))$ of the stabilizer of a.
We want to capture the property of being continuous as a homomorphism from $\operatorname{Aut}(\mathcal{M})$ to $\operatorname{Sym}(F(\mathcal{M}))$, for the functor F.

Definition 9.3. Let $F : \mathbf{Mod}(T) \to \mathbf{Set}$ be a functor, $\mathcal{M} \preccurlyeq \mathbb{U}$ a model of T, $a \in F(\mathcal{M})$, and $X \subseteq \mathcal{M}$. Then we say that

- X is a *support* of a under F if for all elementary maps $f, g : \mathcal{M} \to \mathbb{U}$ with $f|_X = g|_X$, we get $F(f)(a) = F(g)(a)$,
- $\mathcal{M}$ has *finite supports* under F if all elements $a \in F(\mathcal{M})$ have finite support.

We will occasionally just say that X is a finite support of a, or that $\mathcal{M}$ has finite supports, when the underlying functor F is clear from the context.

Example 9.4. If $F : \mathbf{Mod}(T) \to \mathbf{Set}$ is the forgetful functor, then $X \subseteq \mathcal{M}$ is a support of $a \in M$, if and only if any elementary embedding $f : \mathcal{M} \to \mathbb{U}$ which fixes X, fixes a. By homogeneity, this is equivalent to $\mathrm{Aut}_X(\mathbb{U}) \subseteq \mathrm{Aut}_{\{a\}}(\mathbb{U})$, or $a \in \mathrm{dcl}(X)$ (see Section 6.2.1). Thus, any $a \in \mathcal{M}$ is a support of itself, so that any model has finite supports under F.

As intended, if a model $\mathcal{M}$ has finite supports under F, this is precisely what it means for F to be continuous on $\mathrm{Aut}(\mathcal{M})$.

Lemma 9.5. *If $F : \boldsymbol{Mod}(T) \to \boldsymbol{Set}$ has finite supports on a model $\mathcal{M} \preccurlyeq \mathbb{U}$, then $F|_{\mathrm{Aut}(\mathcal{M})} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Sym}(F(\mathcal{M}))$ is a continuous homomorphism.*

Proof. $F|_{\mathrm{Aut}(\mathcal{M})}$ is homomorphic since F is a functor.
Now fix some $a \in F(\mathcal{M})$, and find by assumption a finite support $X \subseteq \mathcal{M}$ of a.
Let $f \in \mathrm{Aut}_X(\mathcal{M})$. Then $\iota_{\mathcal{M}} \circ f$ is an elementary map $\mathcal{M} \to \mathbb{U}$ with $(\iota_{\mathcal{M}} \circ f)|_X = \iota_{\mathcal{M}}|_X$, and since X is a support of a,

$$F(\iota_{\mathcal{M}} \circ f)(a) = F(\iota_{\mathcal{M}})(a).$$

Injectivity of $F(\iota_{\mathcal{M}})$ yields $F(f)(a) = a$. By the observations at the beginning of this section, we conclude that $F|_{\mathrm{Aut}(\mathcal{M})}$ is continuous. $\square$

Definition 9.6. We say that $F : \mathbf{Mod}(T) \to \mathbf{Set}$ has the *finite support property* (f.s.p.) if all models $\mathcal{M} \in \mathbf{Mod}(T)$ have finite supports under F.

Remark 9.7. *It is easy to see that if $f : \mathcal{M} \to \mathcal{N}$ is an isomorphism of models of T, and $X \subseteq \mathcal{M}$ is a support of a, then $f(X)$ is a support of $f(a)$. In particular, under the assumption of $\aleph_0$-categoricity of T, all countable models have finite supports, as long as one does.*

In the same way as we did for model-continuity, we naturally define the f.s.p. of a functor $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$, as f.s.p. of $\tilde{F} : \mathbf{Mod}(T) \to \mathbf{Set}$.

Corollary 9.8. *If $F : \boldsymbol{Mod}(T) \to \boldsymbol{Mod}(T')$ has the f.s.p. then for any model $\mathcal{M} \preccurlyeq \mathbb{U}$, the group homomorphism $F|_{\mathrm{Aut}(\mathcal{M})} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(F(\mathcal{M}))$ is continuous.*

Proof. Firstly, $F|_{\mathrm{Aut}(\mathcal{M})}$ is a homomorphism to $\mathrm{Aut}(F(\mathcal{M}))$ by functoriality of F. Since F has finite supports on $\mathcal{M}$, Lemma 9.5 tells us that $\tilde{F}|_{\mathrm{Aut}(\mathcal{M})} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Sym}(F(\mathcal{M}))$ is continuous. $\mathrm{Aut}(F(\mathcal{M}))$ carries the subspace topology of $\mathrm{Sym}(F(\mathcal{M}))$, so it follows that $F|_{\mathrm{Aut}(\mathcal{M})} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(F(\mathcal{M}))$ is continuous. □

9.3 L-Continuous Functors and G-Finite Theories

We prove in this section the central result of the Lascar Reconstruction Theorem, namely that if the theory T is G-finite, then any model-continuous functor $F : \mathbf{Mod}(T) \to \mathbf{Set}$ or $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ has the f.s.p. Given Corollary 9.8, and the MCAZ Theorem, Lascar's Reconstruction Theorem will follow. We will make use in this proof of the work in Chapters 6, 7, and 8.

Recall that we have fixed a saturated model $\mathbb{U}$ of regular cardinality $> 2^{|T|}$, and for any $\mathcal{M} \preccurlyeq \mathbb{U}$, we denote by $\iota_{\mathcal{M}}$ the inclusion map $\mathcal{M} \to \mathbb{U}$. It is not difficult to see that any element of $\mathrm{im}[F(\iota_{\mathcal{M}})]$ is supported by $\mathcal{M}$.

Lemma 9.9. *For a functor $F : \boldsymbol{Mod}(T) \to \boldsymbol{Set}$, any $\mathcal{M} \preccurlyeq \mathbb{U}$ is a support of all $b \in \mathrm{im}[F(\iota_{\mathcal{M}})]$.*

Proof. Let $b = F(\iota_{\mathcal{M}})(a) \in \mathrm{im}[F(\iota_{\mathcal{M}})]$, and $f \in \mathrm{Aut}_{\mathcal{M}}(\mathbb{U})$. Then

$$F(f)(b) = F(f \circ \iota_{\mathcal{M}})(a) = F(\iota_{\mathcal{M}})(a) = b.$$

□

We now define a property we call *L-continuity* for a functor $F : \mathbf{Mod}(T) \to \mathbf{Set}$.

Definition 9.10. We say that a functor $F : \mathbf{Mod}(T) \to \mathbf{Set}$ is *L-continuous* if for each model $\mathcal{M} \preccurlyeq \mathbb{U}$ and $b \in \mathrm{im}[F(\iota_{\mathcal{M}})]$ there exists a finite set $X \subseteq \mathcal{M}$ such that

$$\forall f \in \mathrm{Autf_L}(\mathbb{U}/X) : \; F(f)(b) = b.$$

The following proof shows that all model-continuous functors $F : \mathbf{Mod}(T) \to \mathbf{Set}$ are L-continuous. This is an adaptation of Theorem 712 in [16]. We apply here the two main theorems of Chapter 8, namely Theorems 8.22 and 8.23, as a tool in transitioning from model-completeness to L-continuity.

Theorem 9.11. *Let $F : \boldsymbol{Mod}(T) \to \boldsymbol{Set}$ be a model-continuous functor. Then F is L-continuous.*

Proof. Let $\mathcal{M} \preccurlyeq \mathbb{U}$. Recall that we can write $\mathcal{M}$ as a union $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ where $(\mathcal{M}_i)_{\imath \in I}$ is the family of its countable submodels. Since F is model-continuous, we have

$$\mathrm{im}[F(\iota_{\mathcal{M}})] = \bigcup_{i \in I} \mathrm{im}[F(\iota_{\mathcal{M}_i})].$$

Thus, it is enough to prove that for any countable model $\mathcal{M} \preccurlyeq \mathbb{U}$ and $b \in \mathrm{im}[F(\iota_{\mathcal{M}})]$, there exists a finite set $X \subseteq \mathcal{M}$ with

$$\forall f \in \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}/X): \ F(f)(b) = b.$$

So, let $\mathcal{M} \preccurlyeq \mathbb{U}$ be countable and fix $a \in F(\mathcal{M})$ and $b := F(\iota_{\mathcal{M}})(a)$. Let $\overline{\mathcal{M}} = \{x_0, x_1, \dots\}$ be an enumeration of $\mathcal{M}$, and define for each $n \in \omega$, $X_n := (x_0, \dots, x_n)$. Then by Theorem 8.23, we can find an elementary chain of countable submodels $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \cdots \preccurlyeq \mathbb{U}$ such that $X_n \subseteq \mathcal{M}_n$ and $\mathcal{M} \mathop{\smile\!\!\!\!\!|}^{wL}_{X_n} \mathcal{M}_n$ for each $n \in \omega$. Now $\bigcup_n \mathcal{M}_n =: \mathcal{M}^*$ is itself a countable model, containing $\mathcal{M}$. Denoting by $\subseteq (\mathcal{M}, \mathcal{M}')$ the inclusion $\mathcal{M} \to \mathcal{M}'$, we get

$$b = F(\iota_{\mathcal{M}})(a) = F\big(\iota_{\mathcal{M}^*} \circ \subseteq (\mathcal{M}, \mathcal{M}^*)\big)(a) \ \in \ \mathrm{im}(F(\iota_{\mathcal{M}^*})) = \bigcup_{n \in \omega} \mathrm{im}[F(\iota_{\mathcal{M}_n})],$$

where the last equality holds since F is model-continuous. It follows that there exists $k \in \omega$ such that $b \in \mathrm{im}[F(\iota_{\mathcal{M}_k})]$. Applying Lemma 9.9 to $\mathcal{M}$ and $\mathcal{M}_k$, we conclude that

$$\forall f \in \mathrm{Aut}_{\mathcal{M}_k}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}): \ F(f)(b) = b.$$

Moreover, since $\mathcal{M} \mathop{\smile\!\!\!\!\!|}^{wL}_{X_k} \mathcal{M}_k$, Theorem 8.22 tells us that

$$\mathrm{Autf}_{\mathrm{L}}(\mathbb{U}/X_k) = \big\langle \, \mathrm{Aut}_{\mathcal{M}}(\mathbb{U}) \cup \mathrm{Aut}_{\mathcal{M}_k}(\mathbb{U}) \big\rangle$$

so we conclude that for any $f \in \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}/X_k)$, we get $F(f)(b) = b$. Since $X_k = \{x_0, \dots, x_k\}$ is finite, we are done. $\square$

Loosely speaking, under an L-continuous functor, all models have something close to finite supports. If $\mathcal{M} \preccurlyeq \mathbb{U}$ has the additional property

$$\forall X \subseteq \mathcal{M} \text{ finite } \exists Y \subseteq \mathcal{M} \text{ finite}: \ \mathrm{Aut}_Y(\mathbb{U}) \subseteq \mathrm{Autf}_{\mathrm{L}}(\mathbb{U}/X) \tag{9.1}$$

then $\mathcal{M}$ in fact has finite supports under any L-continuous functor.

Lemma 9.12. *Let $F : \boldsymbol{Mod}(T) \to \boldsymbol{Set}$ be L-continuous. It $\mathcal{M} \preccurlyeq \mathbb{U}$ has the property (9.1), then $\mathcal{M}$ has finite supports under F.*

Proof. Fix $a \in F(\mathcal{M})$, and let $b := F(\iota_{\mathcal{M}})(a)$. By L-continuity, find a finite set $X \subseteq \mathcal{M}$ such that

$$\forall f \in \mathrm{Autf_L}(\mathbb{U}/X) : \ F(f)(b) = b.$$

By property (9.1), find a finite set $Y \subseteq \mathcal{M}$ with $\mathrm{Aut}_Y(\mathbb{U}) \subseteq \mathrm{Autf_L}(\mathbb{U})$. Then

$$\forall f \in \mathrm{Aut}_Y(\mathbb{U}) : \ F(f)(b) = b. \tag{9.2}$$

Recall that $b = F(\iota_{\mathcal{M}})(a)$, and let $g, h : \mathcal{M} \to \mathbb{U}$ be elementary maps with $g|_Y = h|_Y$. Extend, by homogeneity, g and h to $g', h' \in \mathrm{Aut}(\mathbb{U})$. Then $g'^{-1}h' \in \mathrm{Aut}_Y(\mathbb{U})$, so

$$F(g)(a) = F(g' \circ \iota_{\mathcal{M}})(a) = F(g')(b) = F(h')(b) = F(h' \circ \iota_{\mathcal{M}})(a) = F(h)(a).$$

We conclude that Y is a support of a. Since $Y \subseteq \mathcal{M}$ and Y is finite, we are done. □

If we apply this Lemma to Theorem 9.11, we immediately obtain the following.

Corollary 9.13. *Let $F : \boldsymbol{Mod}(T) \to \boldsymbol{Set}$ be a model-continuous functor, and $\mathcal{M} \preccurlyeq \mathbb{U}$ a model with property (9.1). Then $\mathcal{M}$ has finite supports under F.*

In case of a G-finite, $\aleph_0$-categorical theory T, the final result of Chapter 7 (Corollary 7.18) tells us precisely that every model $\mathcal{M} \preccurlyeq \mathbb{U}$ has the property (9.1). Hence, in this case, we immediately see that model-continuous functors in fact have the f.s.p. This is Corollary 714 of [16].

Corollary 9.14. *If T is $\aleph_0$-categorical and G-finite, then any model-continuous functor $F : \boldsymbol{Mod}(T) \to \boldsymbol{Set}$ has the f.s.p.*

Proof. Follows immediately from Corollary 7.18 and Corollary 9.13. □

If we remove the assumption of $\aleph_0$-categoricity, but assume that T is *G-trivial* - that is, has trivial Lascar group $\mathrm{Gal_L}(\mathbb{U}/X)$ over all finite sets X, then the above result still holds.

Corollary 9.15. *If T is G-trivial, then any model-continuous functor $F : \boldsymbol{Mod}(T) \to \boldsymbol{Set}$ has the f.s.p.*

Proof. For any finite $X \subseteq \mathbb{U}$, G-triviality yields $\mathrm{Autf_L}(\mathbb{U}/X) = \mathrm{Aut}(\mathbb{U})$, so any model $\mathcal{M} \preccurlyeq \mathbb{U}$ trivially satisfies (9.1). The statement then follows from Corollary 9.13. □

We have seen in Section 9.1 that equivalences $F : \mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ are model-continuous, so by what we have just shown, they have the f.s.p. By the results of Section 9.2, this means that the induced maps $\mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(F(\mathcal{M}))$ are continuous. Combining all of this, the MCAZ Theorem tells us that $\mathbf{Mod}(T)$ determines T up to bi-interpretability.

Theorem 9.16. *Let T be an $\aleph_0$-categorical and G-finite theory, and T' a countable theory. If $\textbf{\textit{Mod}}(T) \simeq \textbf{\textit{Mod}}(T')$, then T and T' are bi-interpretable.*

Proof. Let T and T' be given as in the statement. Note first that since $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$, Corollary 4.9 tells us that T' is $\aleph_0$-categorical as well.
Let $F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ be an equivalence of categories. Then F is model-continuous by Lemma 9.2. Equivalently, $\tilde{F} : \mathbf{Mod}(T) \to \mathbf{Set}$ is model-continuous, so by Corollary 9.14, F has the f.s.p. Fixing a countable model $\mathcal{M}$ of T, $\mathcal{M}' := F(\mathcal{M})$ is countable by Corollary 4.9. Moreover, Corollary 9.8 tells us that the restriction

$$\gamma := F|_{\mathrm{Aut}(\mathcal{M})} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$$

is continuous w.r.t. the topologies of pointwise convergence. Since F is a functor, γ is a group-homomorphism and since F is fully faithful, γ is bijective. Note that $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{M}')$ are Polish (i.e. separable and completely metrizable) since $\mathcal{M}$ and $\mathcal{M}'$ are countable. Thus, by an open mapping theorem for polish groups (see for example [11]), γ is in fact a homeomorphism. Now, the MCAZ Theorem (5.11) tells us precisely that T and T' are bi-interpretable. $\square$

Remark 9.17. *For the proof of Theorem 9.16, it would suffice for some countable model of T to have finite supports under the equivalence $F : \textbf{\textit{Mod}}(T) \to \textbf{\textit{Mod}}(T')$, rather than all models. However, under $\aleph_0$-categoricity, if one countable model has finite supports, then all do. Given that F is model-continuous and we can write any model of T as the union of its countable submodels, this already amounts to F having the f.s.p.*

If we additionally assume $\aleph_0$-categoricity of T', recall that by Theorem 4.11, any equivalence $\mathbf{End}(\mathcal{M}_T) \simeq \mathbf{End}(\mathcal{M}_{T'})$ of the endomorphism monoids of the countable models induces an equivalence $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$. We can therefore strengthen the reconstruction theorem slightly.

Corollary 9.18. *Let T and T' be $\aleph_0$-categorical theories, and T G-finite. If $\textbf{\textit{End}}(\mathcal{M}_T) \simeq \textbf{\textit{End}}(\mathcal{M}_{T'})$, then T and T' are bi-interpretable.*

Proof. Follows immediately from Theorem 4.11 and Theorem 9.16. $\square$

9.4 Concluding Remarks

The Reconstruction Theorem is stated and proved in Lascar [16] in more categorical terms. To each $\mathcal{L}$-theory T, one can associate a category $\mathbf{Def}(T)$, called the *category of definable sets*, whose objects are $\mathcal{L}$-formulas modulo equivalence under T, and morphisms are formulas which T proves to be functions between the respective definable sets. Makkai and Reyes [19] show that one can 'complete' $\mathbf{Def}(T)$ to a *pretopos*. It turns out that this so-called *pretopos-completion* corresponds precisely to Shelah's $(-)^{eq}$-construction (see Section 2.5), i.e. the pretopos completion of $\mathbf{Def}(T)$ is just $\mathbf{Def}(T^{eq})$. Now, the original reconstruction theorem in Lascar [16] states that the category $\mathbf{Mod}(T)$ uniquely determines $\mathbf{Def}(T^{eq})$, up to equivalence. This can be seen as a direct corollary of the version of the reconstruction theorem as we state it, given the fact that - still assuming $\aleph_0$-categoricity - T and T^{eq} are bi-interpretable, and that bi-interpretations $T^{eq} \simeq (T')^{eq}$ correspond to equivalences $\mathbf{Def}(T^{eq}) \simeq \mathbf{Def}((T')^{eq})$ [8, Chapter 4]. The proof in [16] does not make use of the MCAZ Theorem as formulated in Section 5.2, but instead, a result by Makkai [18, Theorem 3.2]: The evaluation functor $ev : \mathbf{Def}(T^{eq}) \to \mathbf{Set}^{\mathbf{End}(\mathcal{M}_T)}$, which maps a formula φ to the functor

$$ev(\varphi) : \begin{cases} \mathbf{End}(\mathcal{M}_T) \to \mathbf{Set} \\ \mathcal{M}_T \mapsto \varphi(\mathcal{M}_T) \\ f \mapsto f|_{\varphi(\mathcal{M}_T)}. \end{cases}$$

is fully faithful, and any functor $\mathbf{End}(\mathcal{M}_T) \to \mathbf{Set}$ has the f.s.p. if and only if it is isomorphic to a functor of the form $ev(\varphi)$.

The advantage of our approach is that - apart from working with the notion of functors between categories of models or sets - we are able to prove the Reconstruction Theorem in a more model-theoretic language, and identify the relevant part of [18, Theorem 3.2] as the MCAZ Theorem.

It is worth noting that Makkai [18] mentions already the likely relevance of model-continuous functors (which he calls *up-continuous*) in reconstructing theories from their categories of models, and credits André Joyal for this suggestion.

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