Notes on a reconstruction theorem of Lascar

Reconstructing an ω-categorical structure from its monoid of elementary embeddings.

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Theorem (Makkai 1982 and Coquand in Ahlbrandt, Ziegler 1986)

$\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(\mathcal{B})$ *as topological groups with* τ_{pw} *iff* $\mathcal{A}$ *and* $\mathcal{B}$ *are bi-interpretable.*

Here τ_{pw} denotes the topology of pointwise convergence on $\mathrm{Aut}(\mathcal{A})$, respectively $\mathrm{Aut}(\mathcal{B})$.

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- Yes, if $\mathcal{A}$ is *G-finite* (Lascar 1982).

Lascar's reconstruction theorem

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Let $\mathcal{A}$ and $\mathcal{B}$ be countable and ω-categorical. Suppose $\mathcal{A}$ is G-finite. If there is a monoid isomorphism $F : \mathrm{EEmb}(\mathcal{A}) \to \mathrm{EEmb}(\mathcal{B})$, then $\mathcal{A}$ and $\mathcal{B}$ are bi-interpretable.

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For finite $X \subseteq \mathcal{A}$, write $\mathrm{Autf}(\mathcal{A}/X)$ for the subgroup of $\mathrm{Aut}_X(\mathcal{A})$ generated by the set

$$\left\{ f \in \mathrm{Aut}_X(\mathcal{A}) \;\middle|\; \exists \mathcal{M} \succ \mathcal{A},\ X \subseteq \mathcal{A}' \prec \mathcal{M}, f' \in \mathrm{Aut}(\mathcal{M}) :\ f \subseteq f',\ f' \in \mathrm{Aut}_{\mathcal{A}'}(\mathcal{M}) \right\}.$$

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The central step of Lascar's Reconstruction Theorem is as follows.

Lemma (Lascar 1982)

Let $F : \mathrm{EEmb}(\mathcal{A}) \to \mathrm{EEmb}(\mathcal{B})$ *be an isomorphism of monoids. Then the restriction* $\tilde{F} : \big(\mathrm{Aut}(\mathcal{A}), \tau_L\big) \to \big(\mathrm{Aut}(\mathcal{B}), \tau_{pw}\big)$ *is continuous.*

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Bi-interpretability follows by Ahlbrandt–Ziegler–Coquand–Makkai.

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Note: This only uses saturation of $\mathcal{A}$, not ω-categoricity.

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