Reconstruction of ω-categorical structures and the Lascar group

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Elementary Maps

$\mathcal{L}\ldots$ countable first-order language (e.g. $\mathcal{L} = \{\leq, +, \cdot, 0, 1\}$)
$\mathcal{M}, \mathcal{N}\ldots$ infinite $\mathcal{L}$-structures (e.g. $\mathcal{M} = \langle \mathbb{Z}; \leq, +, \cdot, 0, 1\rangle$)

A map $f : M \to N$ is called *elementary* if for any $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n)$,

$$\forall \overline{x} \in M^n : \ \mathcal{M} \models \varphi(\overline{x}) \iff \mathcal{N} \models \varphi(f(\overline{x})).$$

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If $\mathcal{M} \subseteq \mathcal{N}$ and $\subseteq : \mathcal{M} \to \mathcal{N}$ is elementary, we write $\mathcal{M} \preccurlyeq \mathcal{N}$.

We call the monoid $\mathrm{End}(\mathcal{M})$ of elementary maps $\mathcal{M} \to \mathcal{M}$ the **endomorphism monoid** of $\mathcal{M}$.
We call the group $\mathrm{Aut}(\mathcal{M})$ of automorphisms of $\mathcal{M}$ the **automorphism group** of $\mathcal{M}$.

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Q: How much do these automorphism groups tell us about $\mathcal{M}$?

In general, not much. $\mathrm{Aut}(\langle \mathbb{N}, \leq \rangle) = \mathrm{Aut}(\langle \mathbb{N}, (c_n)_{n \in \mathbb{N}} \rangle) = \{\mathrm{id}\}$.

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From here onward let T be a complete $\mathcal{L}$-theory with infinite models.

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The theory $\mathrm{Th}(\langle \mathbb{N}, = \rangle)$ of infinite sets ;
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Let $\mathcal{M}$ be a countable structure. For $\overline{x} \in M^n$, the *orbit* of $\overline{x}$ under $\operatorname{Aut}(\mathcal{M})$ is

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$\mathcal{M}$ is ω-categorical iff $M^n / \mathrm{Aut}(\mathcal{M})$ is finite for all $n \in \mathbb{N}$.
Moreover, in this case a set $X \subseteq M^n$ is definable iff $\mathrm{Aut}(\mathcal{M}) \cdot X = X$.

Reconstruction

Let $\mathcal{M}, \mathcal{M}'$ be countable ω-categorical structures in languages $\mathcal{L}$, $\mathcal{L}'$.

Corollary

$\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{M}')$ *as permutation groups iff* $\mathcal{M}$ *and* $\mathcal{M}'$ *are bi-definable.*

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Example

For A infinite and $c, d \in A$, $\langle A, c \rangle$ and $\langle A, c, d \rangle$ are bi-interpretable but not bi-definable.

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- Yes, if $\mathcal{M}$, $\mathcal{M}'$ have the *small index property*, i.e. open subgroups of $\mathrm{Aut}(\mathcal{M}_i)$ are precisely those with countable index.

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- In general no (Bordirsky et al. 2016 [2])
- Yes, if $\mathcal{M}, \mathcal{M}'$ are *G-finite* (Lascar '82 [4]).

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Write $\mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$ for the subgroup

$$\Big\langle \bigcup_{\mathcal{M} \prec \mathbb{U}} \mathrm{Aut}(\mathbb{U}/\mathcal{M}) \Big\rangle \trianglelefteq \mathrm{Aut}(\mathbb{U})$$

of **Lascar strong** automorphisms of $\mathbb{U}$. We call $\mathrm{Gal}_{\mathrm{L}}(\mathbb{U}) := \mathrm{Aut}(\mathbb{U}) / \mathrm{Autf}_{\mathrm{L}}(\mathbb{U})$ the **Lascar group** of $\mathbb{U}$.

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For $X \subseteq \mathbb{U}$, write $\operatorname{Autf}_{\mathsf{L}}(\mathbb{U}/X)$ for $\operatorname{Autf}_{\mathsf{L}}(\mathbb{U}_X) \leq \operatorname{Aut}(\mathbb{U}/X)$ and call $\operatorname{Gal}_{\mathsf{L}}(\mathbb{U}/X) := \operatorname{Aut}(\mathbb{U}/X) / \operatorname{Autf}_{\mathsf{L}}(\mathbb{U}/X)$ the Lascar group of $\mathbb{U}$ over X.

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Examples

For A an uncountable set and $X \subseteq A$ countable $\mathrm{Gal}_{\mathsf{L}}(\underline{A/X}) = \{1\}$.
For $\mathbb{C} = \langle \mathbb{C}, +, \cdot, 0, 1 \rangle$ and $X \subseteq \mathbb{C}$, $\mathrm{Gal}_{\mathsf{L}}(\mathbb{C}/X) = \mathrm{Gal}(\overline{\mathbb{Q}(X)}/\mathbb{Q}(X))$.

G-finiteness

It can be shown that $\mathrm{Gal}_{\mathsf{L}}(\mathbb{U}/X)$ does not depend on $\mathbb{U}$, so the following definition is sensible.

Definition

T is *G-finite* if $\mathrm{Gal}_{\mathsf{L}}(\mathbb{U}/X)$ is finite for any finite X.

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Lemma (Lascar [4])

If T is G-finite and $\mathcal{M} \prec \mathbb{U}$ then for finite $X \subseteq M$ there is some finite $Y \subseteq M$ with

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Setting $\mathrm{Autf}_\mathrm{L}(\mathcal{M}/X) := \{f \in \mathrm{Aut}(\mathcal{M}) \mid \exists g \in \mathrm{Autf}_\mathrm{L}(\mathbb{U}/X) : \; f \subseteq g\}$, *we get*

$$\mathrm{Aut}(\mathcal{M}/Y) \subseteq \mathrm{Autf}_\mathrm{L}(\mathcal{M}/X).$$

Lascar's Reconstruction Theorem

Theorem (Lascar '82 [4])

Let $\mathcal{M}$, $\mathcal{M}'$ be countable ω-categorical structures in languages $\mathcal{L}$, $\mathcal{L}'$, and suppose $\mathrm{Th}(\mathcal{M})$ *is G-finite. If* $\mathrm{End}(\mathcal{M}) \cong \mathrm{End}(\mathcal{M}')$ *as monoids, then $\mathcal{M}$ and $\mathcal{M}'$ are bi-interpretable.*

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Proof Sketch:

Let $F : \mathrm{End}(\mathcal{M}) \to \mathrm{End}(\mathcal{M}')$ be an isomorphism of monoids. Note that the (co)restriction $\tilde{F} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$ is a group isomorphism. If $\tilde{F}$ is homeomorphic, then by Coquand-Ahlbrand-Ziegler, we are done. By an open mapping theorem for Polish groups, it is enough to show that $\tilde{F}$ is continuous.

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<u>Step 1:</u> Extend F to a functor $F : \mathrm{Mod}(\mathrm{Th}(\mathcal{M})) \to \mathrm{Mod}(\mathrm{Th}(\mathcal{M}'))$ between the categories of models with elementary maps between them such that F preserves limits of elementary chains of countable models $\mathcal{N}_1 \prec \mathcal{N}_2 \prec \mathcal{N}_3 \prec \dots$, i.e.

$$"F(\bigcup_{i\in\mathbb{N}} \mathcal{N}_i) = \bigcup_{i\in\mathbb{N}} F(\mathcal{N}_i)".$$

Lascar's Reconstruction Theorem

Fix an uncountable saturated $\mathbb{U} \succ \mathcal{M}$ and write $\subseteq$ for the embedding $\mathcal{M} \to \mathbb{U}$.

Step 2: Show that F is *L-continuous*, i.e. for any $b \in \mathcal{M}' = F(\mathcal{M})$ there exists finite $X \subseteq \mathcal{M}$ such that

$$\forall f \in \mathrm{Autf_L}(\mathbb{U}/X) : F(f|_{\mathcal{M}})(b) = F(\subseteq)(b).$$

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Step 3: By G-finiteness of $\mathcal{M}$, we can find finite $Y \subseteq \mathcal{M}$ such that $\mathrm{Aut}(\mathbb{U}/Y) \subseteq \mathrm{Autf_L}(\mathbb{U}/X)$, so for each $b \in \mathcal{M}'$ there is finite $Y \subseteq \mathcal{M}$

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Step 3: By G-finiteness of $\mathrm{Th}(\mathcal{M})$, we can find finite $X' \subseteq \mathcal{M}$ such that $\mathrm{Aut}(\mathbb{U}/Y) \subseteq \mathrm{Autf}_{\mathsf{L}}(\mathbb{U}/X)$, so for each $b \in \mathcal{M}'$ there is finite $X' \subseteq \mathcal{M}$

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Then for any $g \in \mathrm{Aut}(\mathcal{M}/Y)$, there is $f \in \mathrm{Aut}(\mathbb{U}/Y)$ extending g and we get

$$F(\subseteq \circ g)(b) = F(f|_{\mathcal{M}})(b) = F(\subseteq)(b)$$

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so by injectivity of $F(\subseteq)$, we get $F(g)(b) = b$.
We conclude that $F(\mathrm{Aut}(\mathcal{M}/Y)) \subseteq \mathrm{Aut}(\mathcal{M}'/b)$, so $\tilde{F} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}')$ is continuous.

$\square$

[1] Gisela Ahlbrandt and Martin Ziegler. "Quasi-finitely axiomatizable totally categorical theories". In: *Ann. Pure Appl. Logic* 30.1 (1986), pp. 63–82. ISSN: 0168-0072,1873-2461.

[2] Manuel Bodirsky et al. "A counterexample to the reconstruction of ω-categorical structures from their endomorphism monoid". In: *Israel J. Math.* 224.1 (2018), pp. 57–82. ISSN: 0021-2172,1565-8511.

[3] David M. Evans and P. R. Hewitt. "Counterexamples to a conjecture on relative categoricity". In: *Ann. Pure Appl. Logic* 46.2 (1990), pp. 201–209. ISSN: 0168-0072,1873-2461.

[4] Daniel Lascar. "On the category of models of a complete theory". In: *J. Symbolic Logic* 47.2 (1982), pp. 249–266. ISSN: 0022-4812,1943-5886.