The Joys of Completeness

(An invitation to first-order logic)

SJC Mathematical Sciences Graduate Seminar

Mira Tartarotti

Duality of first-order logic

Syntax

Semantics

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Languages ... $L = \{+, 0, <\}$

Semantics

Structures ... $(\mathbb{N}; +^{\mathbb{N}}, 0^{\mathbb{N}}, <^{\mathbb{N}})$

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Duality of first-order logic

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$\overset{?}{\Longleftrightarrow}$

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Completeness +  gives

<u>COROLLARY (Compactness)</u>: Let T be a theory such that $\forall$ finite $T_0 \subseteq T$: $\mathrm{Mod}(T_0) \neq \emptyset$. Then $\mathrm{Mod}(T) \neq \emptyset$.

SOME APPLICATIONS

1. Graph Coloring

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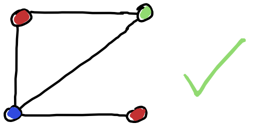

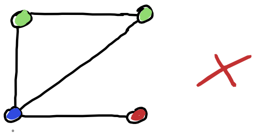

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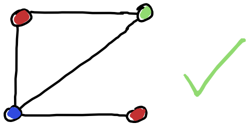

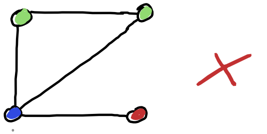

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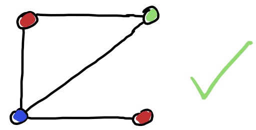

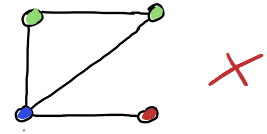

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THM (Tarski): Any statement is true in $\mathbb{C}$ iff it is true in $\overline{\mathbb{F}_p}$ for large p.

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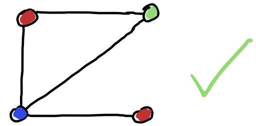

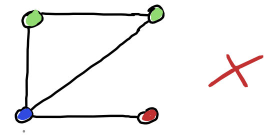

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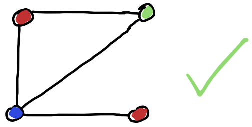

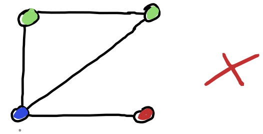

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3. Non-standard analysis (Robinson)

There are models of the theory of $(\mathbb{R}; +, \cdot, 0, 1, <)$ with infinitesimals. (i.e. $\varepsilon > 0$ but $\varepsilon < \frac{1}{n}$ $\forall n \in \mathbb{N}$.)